Some problems on noncommutative harmonique analysis

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Quelques problèmes en analyse harmonique non commutative

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Résumé

Résumé

Cette thèse présente quelques résultats de la théorie des probabilités quantiques et de l'analyse harmonique non commutative. Elle est constituée de trois parties. La première partie démontre l'analogue non commutatif de l'inégalité de John-Nirenberg et la décomposition atomique pour les martingales non commutatives. Ces résultats étendent et améliorent ceux qui existent déjà, et correspondent exactement à ceux que l'on connaît dans le cas classique. La deuxième partie est consacrée à l'étude des espaces de Hardy à valeurs opérateurs via la méthode d'ondelettes. Il est montré que les espaces de Hardy définis par ondelettes coïncident avec ceux définis par les fonctions carrées de Littlewood-Paley et Lusin. Cette approche est similaire à celle du cas des martingales non commutatives, mais l'utilisation des outils de martingales en analyse harmonique permet une démonstration plus rapide. Dans la troisième partie, nous nous tournons vers des applications de la théorie bien établie des espaces de Hardy, c'est-à-dire des opérateurs de Calderón-Zygmund (OCZ pour abréviation) associés à des noyaux à valeurs matricielles. On obtient des estimations de type faible $(1,1)$ pour des OCZ dyadiques parfaites et des shifts de Haar annulateurs associés à des noyaux non commutatifs, ainsi que des estimations de type $H_1 \to L_1$ pour des OCZ arbitaires d'après une décomposition d'une fonction en ligne/colonne. En conjonction avec $L_\infty \to BMO$, nous établissons certaines estimations de type L_p. Cette approche s'applique aussi à des paraproduits et des transformées de martingales avec des symboles et coefficients non commutatifs respectivement.

Mots-clefs

Algèbres de von Neumann, espaces L_p non commutatifs, martingales non commutatives, inégalité de John-Nirenberg, décomposition atomique, espaces de Hardy et BMO à valeurs matricielles, ondelettes, opérateurs de Calderón-Zygmund, noyaux à valeurs matricielles, shifts de Haar, transformées de martingale, paraproduits.

Some problems on noncommutative harmonique analysis

Abstract

This thesis presents some results in the theory of quantum probability and noncommutative harmonic analysis. It consists of three parts. The first part presents the noncommutative analogue of the John-Nirenberg inequality and atomic decomposition for the noncommutative martingales. These results extend and improve the existing ones, and correspond exactly to those in the classical case. The second part is devoted to the study of operator-valued Hardy spaces via wavelet method. It is shown that Hardy spaces defined by wavelets coincide with those defined through the usual Lusin and Littlewood-Paley square functions. This approach is parallel to that in the noncommutative martingale case, hence much concise. In the third part, we turn to applications of the well established theory of Hardy spaces, i.e. Calderón-Zygmund operators (CZO for abbreviation) associated to matrix-valued kernels. We obtain weak $(1,1)$ type estimates for perfect dyadic CZO's and cancellative Haar shift associated to noncommuting kernels, and $H_1 \to L_1$ type estimates for arbitrary CZO's in terms of a row/column decomposition of the functions. In conjunction with $L_\infty \to BMO$, we get certain row/column L_p estimates. The approach also applies to paraproducts and martingales transforms with noncommuting symbols and coefficients respectively.

Keywords

Von Neumann algebras, noncommutative L_p-spaces, noncommutative martingales, John-Nirenberg inequality, atomic decomposition, matrix-valued Hardy spaces and BMO spaces, wavelets, Calderón-Zygmund operators, matrix-valued kernels, Haar shifts, martingales transforms, paraproducts.

Table des matières

Introduction

La théorie des probabilités quantiques et l'analyse harmonique non commutative se posent dans le cadre des algèbres de von Neumann. L'algèbre de von Neumann est le cadre naturel pour la théorie d'intégration non commutative, où les fonctions de la théorie de l'intégration classique sont remplacées par des opérateurs sur un espace de Hilbert, et les mesure par des traces. Historiquement, c'est dans le but d'étudier la mécanique quantique que von Neumann et ses collaborateurs ont posé les bases de la théorie de l'intégration non commutative. Pour cette raison, l'étude des thèmes dans l'analyse non commutative était sans surprise partiellement motivée par la mécanique quantique. Néanmoins, la théorie des probabilités quantiques et l'analyse harmonique non commutative sont devenues les domaines de recherche indépendants en mathématique.

La théorie des inégalités de martingales non commutatives est importante en probabilités quantiques. Le développement moderne des inégalités de martingales non commutatives a commencé avec le papier fondateur de Pisier et Xu [48] dans lequel les inégalités de Burkholder-Gundy et le théorème de dualité de Fefferman ont été étendus au cas non commutatif. Depuis, de nombreux résultats classiques ont été tranférés avec succès dans le monde non commutatif. Il doit être souligné qu'étendre les résultats classiques au cadre non commutatif exige souvent d'attaquer le problème sous un autre angle. Par exemple, l'argument de temps d'arrêt et la fonction maximale ponctuelle, qui sont souvent utilisés dans les preuves classiques, n'existent pas dans ce cadre. Par conséquent, des techniques ou théories supplémentaires, comme par exemple la théorie des espaces d'opérateurs, sont exploitées afin de traiter les martingales non commutatives. De plus, ces techniques peuvent donner de nouveaux résultats même dans la théorie classique, comme illustré dans [25].

Il est bien connu qu'il existe de nombreuses interactions entre la théorie des probabilités classiques et l'analyse harmonique. Ces interactions sont encore fructueuses et jouent un rôle important dans le cadre non commutatif. Ainsi, le deuxième thème de ce travail concerne l'application de la théorie des martingales non commutatives à l'analyse harmonique non commutative. Motivé par les inégalités de martingales non commutatives, Mei [34] a effectué une étude systématique des espaces de Hardy de fonctions à valeurs dans des espaces L_p non commutatifs. Ses espaces de Hardy non commutatifs sont définis par la fonction intégrale de Lusin non commutative, et sont étroitement liés à ceux définis via le semigroupe de Poisson. Ensuite, en combinant ce lien avec la propriété de dilatation Markovienne des semigroupes d'opérateurs et les propriétés des martingales non commutatives relativement à une filtration continue, obtenus récemment, Junge et Mei [19] et [35] ont établi la théorie des espaces de Hardy associés aux semigroupes d'opérateurs, et ont trouvé quelques applications importantes. Certaines de ces applications sont nouvelles même dans le cadre classique.

Dans la théorie des martingales non commutatives, il existe une technique assez importante : la construction de Cuculescu, qui constitue l'analogue de l'argument de temps d'arrêt, mais sous une forme plus faible que dans le cas classique. Randrianantoanina dans [51] [52] et [53] a utilisé cette construction pour démontrer des estimations de type faible $(1,1)$ des transformées de martingales et des fonctions carrées (conditionnelles). La construction de Cuculescu est aussi un outil clef dans la décomposition de Gundy non commutative démontrée par Parcet et Randrianantoanina [44]. Il est bien connu que la transformée de martingales et la fonction carré correspondent en analyse harmonique à la transformée de Hilbert et la fonction de Littlewood-Paley respectivement. Parcet [43] a établi la décomposition de Calderón-Zygmund non commutative basée sur la construction de Cuculescu, et obtenu les estimations de type faible $(1,1)$ pour les opérateurs de Calderón-Zygmund. Par la suite, les estimations de type faible des fonctions carrées à valeurs opérateurs ont également été obtenues par Parcet et Mei [37]. Une nouvelle propriété découverte par Parcet est un principe de pseudo-localisation, qui est approfondi par Hytönen [13] dans le cas classique.

Cette thèse est constituée de trois parties. Le premier chapitre présente un travail en collaboration avec Mei intitulé “John-Nirenberg inequality and atomic decomposition for noncommutative martingales", qui s'inscrit dans la théorie des martingales non commutatives. Ce travail a été accepté par *J. Funct. Anal.* Le contenu du deuxième chapitre concerne la théorie des espaces de Hardy mentionné dans le troisième paragraphe précédent. Ce chapitre est un travail en collaboration avec Yin intitulé “Wavelet approach to operator-valued Hardy spaces", qui a été accepté par *Revista Mat. Iberoa.* Le dernier chapitre est centré sur les opérateurs de Calderón-Zygmund associés à des noyaux à valeurs matricielles et la transformée de martingales. Il s'agit d'un travail effectué en collaboration avec López, Martell et Parcet intitulé “Calderón-Zygmund operators associated to matrix-valued kernels".

Avant que je détaille chaque chapitre dans la suite de cette introduction, nous rappelons l'objet principal des trois chapitres, c'est-à-dire des espaces L_p non commutatifs. Soit $\mathcal{M}$ une algèbre de von Neumann semifinie munie d'une trace normale et fidèle τ et $S^+_{\mathcal{M}}$ l'ensemble des éléments positifs x de $\mathcal{M}$ tels que $\tau(s(x)) < \infty$, où $s(x)$ est la plus petite projection vérifiant $exe = x$. Soit $S_{\mathcal{M}}$ l'espace vectoriel engendré par $S^+_{\mathcal{M}}$. Alors tout élément $x \in S_{\mathcal{M}}$ a une trace finie, et $S_{\mathcal{M}}$ est une *-sous-algèbre w^*-dense de $\mathcal{M}$. Soit maintenant $0 < p < \infty$. Pour tout $x \in S_{\mathcal{M}}$, l'opérateur $|x|^p$ appartient à $S^+_{\mathcal{M}}$ (où $|x| = (x^*x)^{1/2}$ désigne le module de x). Nous définissons alors

$$\|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}, \qquad \forall x \in S_{\mathcal{M}}.$$

On peut vérifier que $\|\cdot\|_p$ est bien définie et est une (quasi)norme sur $S_{\mathcal{M}}$. Le complété de $(S_{\mathcal{M}}, \|\cdot\|_p)$ est noté $L_p(\mathcal{M})$: c'est l'espace usuel L_p non commutatif associé à $(\mathcal{M}, \tau)$. Pour simplifier les notations, nous écrirons $\mathcal{M}$ à la place de $L_\infty(\mathcal{M})$ munie de la norme d'opérateur $\|\cdot\|_{\mathcal{M}}$. Les éléments de $L_p(\mathcal{M})$ peuvent être décrits comme des opérateurs fermés densément définis sur H, H étant l'espace de Hilbert sur lequel $\mathcal{M}$ agit.

0.1 Chapitre 1

La théorie des inégalités de martingales non commutatives a été développée au cours des dernières années. Nous renvoyons le lecteur, par exemple, à [48], [16], [25], [27] et [51] pour les inégalités de martingales non commutatives, à [40], [2] pour les interpolations des espaces de Hardy et à [44], [45] pour les décompositions de Gundy et Davis non commutatives. Le chapitre 1 est également motivé par les deux travaux suivants. Le premier concerne le théorème de John-Nirenberg non commutatif démontré par Junge et Musat, et le deuxième la décomposition 2-atomique des espaces de Hardy établie par Bekjan, Chen, Perrin et Yin [2].

D'abord, rappelons quelques notions élémentaires des martingales non commutatives. Dans ce chapitre $\mathcal{M}$ est une algèbre de von Neumann finie munie d'une trace normalisée. Soit $(\mathcal{M}_n)_{n\geq 1}$ une filtration croissante de sous-algèbres de von Neumann de $\mathcal{M}$ dont l'union est w^*-dense dans $\mathcal{M}$. Soit $\mathcal{E}_n$ l'espérance conditionnelle de $\mathcal{M}$ relativement à $\mathcal{M}_n$. Une suite $x = (x_n)_{n\geq 1}$ de $L_1(\mathcal{M})$ est une martingale non commutative relativement à $(\mathcal{M}_n)_{n\geq 1}$ si $\mathcal{E}_n(x_{n+1}) = x_n$ pour tout $n \geq 1$. Si de plus tous les x_n sont dans $L_p(\mathcal{M})$ pour un certain $1 \leq p \leq \infty$, alors on dit que x est une martingale L_p. Dans ce cas, on considère

$$\|x\|_p = \sup_{n\geq 1} \|x_n\|_p.$$

Si $\|x\|_p < \infty$, on dit que x est une martingale bornée dans L_p. Soit $x = (x_n)_{n\geq 1}$ une martingale non commutative relativement à $(\mathcal{M}_n)_{n\geq 1}$. On définit $dx_n = x_n - x_{n-1}$ pour $n \geq 1$ avec la convention $x_0 = 0$. La suite $dx = (dx_n)_{n\geq 1}$ est appelée la suite des différences de la martingale x. Dans la suite, pour tout $x \in L_1(\mathcal{M})$ on note $x_n = \mathcal{E}_n(x)$ pour $n \geq 1$. Soit $1 \leq p < \infty$. Définissons $\mathcal{H}_p^c$ (resp. $\mathcal{H}_p^r$) comme le complété de l'ensemble des martingales L_p finies pour la norme $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}_p^r} = \|S_r(x)\|_p$), où $S_c(x)$ et $S_r(x)$ sont définis par

$$S_c(x) = \big(\sum_{k\geq 1} |dx_k|^2\big)^{1/2}, \quad S_r(x) = S_c(x^*).$$

Les espaces de Hardy non commutatifs $\mathcal{H}_p(\mathcal{M})$ sont définis comme suit : si $1 \leq p < 2$, on pose

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M}),$$

et on définit la norme

$$\|x\|_{\mathcal{H}_p} = \inf_{x=y+z} \{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\}.$$

Si $2 \leq p < \infty$, on pose

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M}),$$

et on munit cet espace de la norme

$$\|x\|_{\mathcal{H}_p} = \max\{\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}\}.$$

L'espace $\mathcal{BMO}^c$ est défini comme

$$\mathcal{BMO}^c(\mathcal{M}) = \{x \in L_1(\mathcal{M}) : \|x\|_{\mathcal{BMO}^c} < \infty\},$$

où

$$\|x\|_{\mathcal{BMO}^c} = \sup_{n\geq 1} \|\mathcal{E}_n|x - x_{n-1}|^2\|_\infty^{1/2},$$

et

$$\mathcal{BMO}^r(\mathcal{M}) = \{x : x^* \in \mathcal{BMO}^c(\mathcal{M})\}.$$

On considère l'espace

$$\mathcal{BMO}(\mathcal{M}) = \mathcal{BMO}^c(\mathcal{M}) \cap \mathcal{BMO}^r(\mathcal{M})$$

muni de la norme

$$\|x\|_{\mathcal{BMO}} = \max\{\|x\|_{\mathcal{BMO}^c}, \|x\|_{\mathcal{BMO}^r}\}.$$

On travaillera avec la version conditionnelle des espaces de Hardy et BMO introduite dans [25]. Soit $x = (x_n)_{n\geq 1}$ une martingale finie dans $L_2(\mathcal{M})$. On pose

$$s_c(x) = \big(\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k|^2\big)^{1/2} \quad \text{et} \quad s_r(x) = s_c(x^*),$$

où par convention $\mathcal{E}_0 = \mathcal{E}_1$. Soit alors $0 < p < \infty$. On définit l'espace $\mathsf{h}_p^c(\mathcal{M})$ (resp. $\mathsf{h}_p^r(\mathcal{M})$) comme le complété de l'ensemble des martingales L_∞ finies pour la (quasi)norme $\|x\|_{\mathsf{h}_p^c} = \|s_c(x)\|_p$ (resp. $\|x\|_{\mathsf{h}_p^r} = \|s_r(x)\|_p$). Soit $\mathsf{h}_p^d(\mathcal{M})$ l'espace des martingales dont la suite des differences de martingale est dans $\ell_p(L_p(\mathcal{M}))$, où $\ell_p(L_p(\mathcal{M}))$ est l'espace des suites $a = (a_n)_{n\geq 1}$ dans $L_p(\mathcal{M})$ telles que

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \big(\sum_{n\geq 1} \|a_n\|_p^p\big)^{1/p} < \infty.$$

La version conditionnelle des espaces de Hardy de martingales non commutatives est définie comme suit : si $0 < p < 2$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M}) + \mathsf{h}_p^d(\mathcal{M})$$

muni de la (quasi)norme

$$\|x\|_{\mathsf{h}_p} = \inf_{x=y+z+w} \{\|y\|_{\mathsf{h}_p^c} + \|z\|_{\mathsf{h}_p^r} + \|w\|_{\mathsf{h}_p^d}\}.$$

Si $2 \leq p < \infty$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M}) \cap \mathsf{h}_p^d(\mathcal{M})$$

muni de la norme

$$\|x\|_{\mathsf{h}_p} = \max\{\|x\|_{\mathsf{h}_p^c}, \|x\|_{\mathsf{h}_p^r}, \|x\|_{\mathsf{h}_p^d}\}.$$

L'espace bmo^c est défini par

$$\mathsf{bmo}^c(\mathcal{M}) = \{x \in L_1(\mathcal{M}) : \|x\|_{\mathsf{bmo}^c} < \infty\}$$

où

$$\|x\|_{\mathsf{bmo}^c} = \max\left\{\|\mathcal{E}_1(x)\|_\infty, \quad \sup_{n\geq 1} \|\mathcal{E}_n|x - x_n|^2\|_\infty^{1/2}\right\}.$$

On considère

$$\mathsf{bmo}^r(\mathcal{M}) = \{x : x^* \in \mathsf{bmo}^c(\mathcal{M})\}.$$

Soit $\mathsf{bmo}^d(\mathcal{M})$ l'espace des martingales dont la suite des differences de martingale est dans $\ell_\infty(L_\infty(\mathcal{M}))$, où $\ell_\infty(L_\infty(\mathcal{M}))$ est l'espace des suites $a = (a_n)_{n\geq 1}$ dans $L_\infty(\mathcal{M})$ telles que

$$\|a\|_{\ell_\infty(L_\infty(\mathcal{M}))} = \sup_n \|a_n\|_\infty < \infty.$$

Notons que $\mathsf{bmo}^d(\mathcal{M}) = \mathsf{h}_\infty^d(\mathcal{M})$. Définissons l'espace

$$\mathsf{bmo}(\mathcal{M}) = \mathsf{bmo}^c(\mathcal{M}) \cap \mathsf{bmo}^r(\mathcal{M}) \cap \mathsf{bmo}^d(\mathcal{M}),$$

muni de la norme

$$\|x\|_{\mathsf{bmo}} = \max\{\|x\|_{\mathsf{bmo}^c}, \|x\|_{\mathsf{bmo}^r}, \|x\|_{\mathsf{bmo}^d}\}.$$

0.1.1 L'inégalité de John-Nirenberg

On commence par rappeler l'inégalité de John-Nirenberg de la théorie classique des martingales. Soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace probabilisé et $(\mathcal{F}_n)_{n\geq 1}$ une suite croissante de sous-σ-algèbres de $\mathcal{F}$. On notera $\mathbb{E}_n$ les espérances conditionnelles associées. L'espace $BMO(\Omega)$ est défini comme l'ensemble des fonctions $x \in L_1(\Omega)$ telles que

$$\|x\|_{BMO} = \sup_n \|\mathbb{E}_n|x - x_{n-1}|\|_\infty < \infty.$$

Le théorème de John-Nirenberg classique dit qu'il existe deux constantes universelles c_1, $c_2 > 0$ telles que si $\|x\|_{BMO} < c_2$, alors

$$\sup_n \|\mathbb{E}_n(e^{c_1|x-x_{n-1}|})\|_\infty < 1. \tag{0.1.1}$$

Ce résultat est équivalent à la propriété suivante : pour tout $n \geq 1$, $E \in \mathcal{F}_n$ et $\lambda > 0$, il existe une constante universelle $c > 0$ telle que

$$\frac{1}{\mathbb{P}(E)} \mathbb{P}\big(\{\omega \in E \,:\, |x(\omega) - x_{n-1}(\omega)| > \lambda\}\big) \leq c_2 \exp(-c\lambda/\|x\|_{BMO}). \tag{0.1.2}$$

Il y a encore une caractérisation équivalente : il existe une constante universelle $c > 0$ telle que pour tout $1 \leq p < \infty$,

$$\|x\|_{BMO} \leq \sup_n \sup_{E\in\mathcal{F}_n} \frac{1}{\mathbb{P}(E)^{1/p}} \|(x - x_{n-1})\mathbb{1}_E\|_p \leq cp\|x\|_{BMO}. \tag{0.1.3}$$

Junge et Musat dans [23] ont démontré une version non commutative du théorème de John-Nirenberg similaire à (0.1.3), en prouvant qu'il existe une constante universelle $c > 0$ telle que pour tout $2 \leq p < \infty$,

$$\|x\|_{\mathcal{BMO}} \leq \mathcal{B}_p(x) \leq cp\|x\|_{\mathcal{BMO}},$$

où

$$\begin{aligned}\mathcal{B}_p(x) = \max\{ &\sup_n \sup_{b\in\mathcal{M}_n, \|b\|_p\leq 1} \|(x - x_{n-1})b\|_p, \\ &\sup_n \sup_{b\in\mathcal{M}_n, \|b\|_p\leq 1,} \|b(x - x_{n-1})\|_p\}.\end{aligned}$$

Cependant, ce théorème n'est plus vrai (voir Remark 2.14 pour un contre-example) si on considère $\mathcal{BMO}^c(\mathcal{M})$ et $\mathcal{BMO}^r(\mathcal{M})$ séparément. D'autre part, il ne correspond pas à la forme utilisée couramment de l'inégalité de John-Nirenberg classique. Le premier but de ce chapitre est de remédier à ces aspects du théorème de Junge and Musat. Le théorème suivant est un de nos principaux résultats. Dans ce chapitre, $\mathcal{P}(\mathcal{M})$ dénote l'ensemble des projections de $\mathcal{M}$.

Théorème 0.1.1. *Pour tout $0 < p < \infty$,*

$$\alpha_p^{-1}\|x\|_{\mathsf{bmo}^c} \leq \|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} \leq \beta_p\|x\|_{\mathsf{bmo}^c},$$

où

$$\|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} = \max\big\{\|\mathcal{E}_1(x)\|_\infty, \quad \sup_n \sup_{e\in\mathcal{P}(\mathcal{M}_n)} \frac{1}{(\tau(e))^{1/p}} \|(x - x_n)e\|_{\mathsf{h}^c_p}\big\}.$$

Les deux constantes α_p and β_p ont les propriétés suivantes

(i) $\alpha_p = 1$ *pour* $2 \leq p < \infty$ *;*

(ii) $\alpha_p \leq C^{1/p-1/2}$ *pour* $0 < p < 2$ *;*

(iii) $\beta_p \leq cp$ *pour* $2 \leq p < \infty$ *;*

(iv) $\beta_p = 1$ *pour* $0 < p < 2$.

Un résultat similaire est encore vrai pour $\mathcal{BMO}^c(\mathcal{M})$, mais uniquement pour $2 \leq p < \infty$ (voir Remark 2.9). D'un autre côté, notre preuve du Théorème 0.1.1 peut être modifiée facilement afin d'étendre le résultat de Junge/Musat à tout $0 < p < \infty$ sous la forme (0.1.3) (voir Corollary 2.19). De plus, la constante optimale cp obtenue en utilisant le résultat de Randrianantoanina [52] nous permet de formuler l'inégalité sous la forme (0.1.1) et (0.1.2).

Dans la dernière section de ce chapitre, on donne une réponse négative à une question posée dans [23] (page 136) : étant donné $2 < p < \infty$, existe-t-il une constante c_p telle que

$$\sup_k \|\mathcal{E}_k|x - \mathcal{E}_{k-1}x|^p\|_\infty^{\frac{1}{p}} \leq c_p\|x\|_{\mathcal{BMO}}?$$

Théorème 0.1.2. *Supposons que* $\sup_k \|\mathcal{E}_k|f - \mathcal{E}_{k-1}f|^p\|_\infty^{1/p} \leq c_p(n)\|f\|_{\mathcal{BMO}}$ *pour un certain* $p \geq 3$*. Alors*

$$c_p(n) \geq c(\log(n+1))^{\frac{2}{p}}.$$

0.1.2 Décomposition atomique

Nous nous tournons maintenant vers le deuxième objectif de ce chapitre : la décomposition atomique des espaces de Hardy non commutatifs. Nous rappelons que la décomposition 2-atomique a déjà été obtenue dans [2]. Un élément $a \in L_1(\mathcal{M})$ est un $(1,2)_c$-atome relativement à $(\mathcal{M}_n)_{n\geq 1}$ s'il existe $n \geq 1$ et $e \in \mathcal{P}(\mathcal{M}_n)$ tels que

(i) $\mathcal{E}_n(a) = 0$; (ii)$ae = a$; (iii) $\|a\|_2 \leq (\tau(e))^{-1/2}$.

L'espace de Hardy atomique $\mathsf{h}^c_{1,\mathrm{at}}(\mathcal{M})$ est défini comme l'espace de tous les opérateurs $x \in L_1(\mathcal{M})$ tels que la norme $\|\cdot\|_{\mathsf{h}^c_{1,\mathrm{at}}}$ est finie, où

$$\|x\|_{\mathsf{h}^c_{1,\mathrm{at}}} = \|\mathcal{E}_1 x\|_1 + \inf \sum_j |\lambda_j|.$$

Ici l'infimum est pris sur toutes les décompositions possibles $x - \mathcal{E}_1 x = \sum_j \lambda_j a_j$ telles que $\lambda_j \in \mathbb{C}$, a_j est un $(1,2)_c$-atome. Il est démontré dans [2] que $x \in \mathsf{h}^c_1(\mathcal{M})$ si et seulement si $x \in \mathsf{h}^c_{1,\mathrm{at}}(\mathcal{M})$, avec

$$\|x\|_{\mathsf{h}^c_1} \simeq \|x\|_{\mathsf{h}^c_{1,\mathrm{at}}}.$$

Combiné à l'equivalence $\mathcal{H}^c_1(\mathcal{M}) = \mathsf{h}^c_1(\mathcal{M}) + \mathsf{h}^d_1(\mathcal{M})$, les auteurs de [2] ont aussi obtenu une decomposition 2-atomique pour $\mathcal{H}^c_1(\mathcal{M})$.

Nous rappelons brièvement l'argument utilisé dans [2]. L'espace dual de $\mathrm{h}^c_{1,\mathrm{at}}(\mathcal{M})$ peut être décrit comme

$$\Lambda^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\Lambda^c} < \infty\}$$

avec

$$\|x\|_{\Lambda^c} = \max\{\|\mathcal{E}_1 x\|_\infty, \quad \sup_{n\geq 1} \sup_{e\in\mathcal{P}_n} (\frac{1}{\tau(e)}\tau(e|x - x_n|^2))^{\frac{1}{2}}\}.$$

En fait, le supremum dans la définition précédente peut être pris sur tous les $b \in L_1(\mathcal{M}_n)$, puisque les points extrémaux de la boule unité de $L_1(\mathcal{M}_n)$ sont des multiples de projections. Par conséquent,

$$\begin{aligned} \|x\|_{\Lambda^c} &= \max\{\|\mathcal{E}_1 x\|_\infty, \quad \sup_{n\geq 1}\sup_{b\in\mathcal{M}_n} (\frac{1}{\|b\|_1}\tau(b|x-x_n|^2))^{\frac{1}{2}}\} \qquad (0.1.4)\\ &= \max\{\|\mathcal{E}_1 x\|_\infty, \quad \sup_{n\geq 1}\|\mathcal{E}_n|x-x_n|^2\|_\infty^{\frac{1}{2}}\}\\ &= \|x\|_{\mathsf{bmo}^c}. \end{aligned}$$

Ainsi la dualité $(\mathsf{h}_1^c(\mathcal{M}))^* = \mathsf{bmo}^c(\mathcal{M})$ implique $\mathsf{h}_{1,\mathrm{at}}^c(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M})$.

Il est bien connu dans le cas classique que les 2-atomes dans la décomposition précédente peuvent être remplacés par les q-atomes pour tout $1 < q \leq \infty$. Nous rappelons la définition de ces atomes dans le cadre classique. On dit qu'une fonction $a \in L_1(\Omega)$ est un q-atome s'il existe $n \geq 1$ et $E \in \mathcal{F}_n$ tels que

(i) $\mathbb{E}_n a = 0$; (ii) $\{a \neq 0\} \subset E$; (iii) $\|a\|_q \leq \mathbb{P}(E)^{-1+\frac{1}{q}}$.

Nous nous référons à [57] pour plus d'information.

La principale difficulté pour obtenir la décomposition q-atomique dans le cadre non commutatif est que l'équivalence clef (0.1.4) n'est plus vraie si on remplace dans (iii) l'indice de puissance 2 par $q' \neq 2$, $1 \leq q' < \infty$ où q' désigne l'indice conjugué de q. On surmonte cet obstacle par le Théorème 0.1.1, et on obtient le théorème suivant.

Théorème 0.1.3. *Pour tout* $1 < q \leq \infty$,

$$\mathsf{h}_1^c(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at}_{\mathrm{q,pr}}}^c(\mathcal{M})$$

et ces normes sont équivalentes. Ici, $\mathrm{h}_{1,\mathrm{at}_{\mathrm{q,pr}}}^c(\mathcal{M})$ *est l'espaces de Hardy q-atomique avec ses atomes définis ainsi : a est un q-atome s'il existe $n \geq 1$ et une projection $e \in \mathcal{P}(\mathcal{M}_n)$ tels que*

(i) $\mathcal{E}_n(a) = 0$; (ii) $ae = a$; (iii) $\|a\|_{\mathsf{h}_q^c} \leq (\tau(e))^{-\frac{1}{q'}}$.

Notons que $\mathsf{h}_{1,\mathrm{at}_{2,\mathrm{pr}}}^c(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at}}^c(\mathcal{M})$, et la décomposition 2-atomique est ainsi retrouvée dans [2]. Par ailleurs, en appliquant la version conditionnelle du théorème de Junge et Musat de la forme (0.1.3), on obtient une décomposition q-atomique pour $\mathsf{h}_1(\mathcal{M})$ dans laquelle les atomes sont définis d'une façon similaire, où la norme $\|\cdot\|_{\mathsf{h}_q^c}$ dans (iii) ci-dessus est remplacée par $\|\ \|_q$ et la condition de support (ii) est affaiblie à $r(a) \leq e$ ou $l(a) \leq e$ (voir Theorem 3.19). C'est exactement l'analogue non commutatif de la décomposition atomique classique.

Les inégalités de John-Nirenberg et la décomposition établie ici seront apppliquées pour démontrer les estimations de type $H_1 \to L_1$ dans le chapitre 3.

0.2 Chapitre 2

Motivé par les probabilités quantiques mentionnées dans le chapitre 1 et l'analyse harmonique à valeurs matricielles, Mei dans [34] a défini les espaces H_p pour des fonctions à valeurs opérateurs en considérant la fonction de Lusin à valeurs opérateurs. Pour $1 \leq p < \infty$, Mei définit l'espace $H_p^c(\mathbb{R}, \mathcal{M})$ comme le complété de l'espace des fonctions simples

f à valeurs dans $S_{\mathcal{M}}$ tels que $\|S_c(f)\|_p$ est finie, où $S_c(f)$ est l'analogue non commutatif de l'intégrale de Lusin classique définie par

$$S_c(f)(x) = \Big(\int_\Gamma \Big[\frac{\partial f^*}{\partial t} \frac{\partial f}{\partial t} + \frac{\partial f^*}{\partial y} \frac{\partial f}{\partial y} \Big] (x+y,t)\, dydt \Big)^{\frac{1}{2}}$$

avec $\Gamma = \{(y,t) \in \mathbb{R}^2_+ \mid |y| < t\}$ et $f(y,t) = P_t f(y)$ pour le semi-groupe de Poisson $(P_t)_{t\geq 0}$. Ensuite, $H^r_p(\mathbb{R},\mathcal{M})$ et $H_p(\mathbb{R},\mathcal{M})$ sont définis d'une façon similaire aux espaces analogues dans le cas des martingales.

Le résultat remarquable prouvé par Mei est que $H^c_1(\mathbb{R},\mathcal{M})$ est le prédual de l'espace BMO apparu dans l'analyse harmonique à valeurs matricielles noté $BMO^c(\mathbb{R},\mathcal{M})$. Cet espace est constitué de tous les $\varphi \in L_\infty(\mathcal{M}; L^c_2(\mathbb{R}, dt/1+t^2))$ tels que

$$\|\varphi\|_{BMO^c} = \sup_{I\subset\mathbb{R}} \Big\| \frac{1}{|I|} \int_I |\varphi - \varphi_I|^2 \Big\|^{\frac{1}{2}}_{\mathcal{M}} < \infty.$$

Il a aussi obtenu les résultats d'interpolation désirés et les inégalités de Littlewood-Paley. Mei a adapté l'approche classique dans sa recherche et réduit plusieurs problèmes au cas des martingales. Il a inventé une technique très intéressante et puissante : l'espace BMO sur $\mathbb{R}$ (aussi bien dans les cas classique et non commutatif) est l'intersection de deux espaces BMO dyadiques.

D'un autre côté, en analyse harmonique classique, il est bien connu que $H_1(\mathbb{R})$ défini par la fonction carrée et $\mathcal{H}_1(\mathbb{R})$ défini par ondelettes coïncident puisque'ils admettent la même décomposition atomique. Comme expliqué dans le chapitre 1, il est très difficile d'obtenir la décomposition atomique pour les espaces de Hardy non commutatifs par la construction explicite comme dans le cadre classique. On ne peut donc pas comparer ces deux espaces en utilisant la méthode classique. Dans ce chapitre, on définit directement $\mathcal{H}_p(\mathbb{R},\mathcal{M})$ et $\mathcal{BMO}(\mathbb{R},\mathcal{M})$ via ondelette. Les définitions sont similaires à celles du cas des martingales, mais associées à une base d'ondelettes fixée $(w_I)_{I\in\mathcal{D}}$. Par souci de simplicité, on notera $\mathcal{N}$ l'espace $L_\infty(\mathbb{R})\bar{\otimes}\mathcal{M}$ dans ce chapitre. Pour $f \in S_{\mathcal{N}}$, les fonctions carrées sont définies par

$$S_c(f)(x) = \Big(\sum_{I\in\mathcal{D}} \frac{|\langle f, w_I\rangle|^2}{|I|} \mathbb{1}_I(x) \Big)^{\frac{1}{2}}, \tag{0.2.1}$$

et $S_r(f) = S_c(f^*)$. Les normes sont données par

$$\|f\|_{\mathcal{H}^c_p} = \|S_c(f)\|_{L_p(\mathcal{N})}, \qquad \text{et} \quad \|f\|_{\mathcal{H}^r_p} = \|S_r(f)\|_{L_p(\mathcal{N})}.$$

Puis l'espace $\mathcal{H}^c_p(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}^r_p(\mathbb{R},\mathcal{M})$) est défini comme l'espace complété de $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}^c_p(\mathbb{R},\mathcal{M})})$ (resp. $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}^c_p(\mathbb{R},\mathcal{M})})$). On définit alors les espaces de Hardy à valeurs opérateurs comme suit : pour $1 \leq p < 2$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M}) = \mathcal{H}^c_p(\mathbb{R},\mathcal{M}) + \mathcal{H}^r_p(\mathbb{R},\mathcal{M}) \tag{0.2.2}$$

munis de la norme

$$\|f\|_{\mathcal{H}_p} = \inf\{\|g\|_{\mathcal{H}^c_p} + \|h\|_{\mathcal{H}^r_p} : f = g + h, g \in \mathcal{H}^c_p, h \in \mathcal{H}^r_p\},$$

et pour $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M}) = \mathcal{H}^c_p(\mathbb{R},\mathcal{M}) \cap \mathcal{H}^r_p(\mathbb{R},\mathcal{M}) \tag{0.2.3}$$

munis de la norme

$$\|f\|_{\mathcal{H}_p} = \max\{\|f\|_{\mathcal{H}^c_p}, \|f\|_{\mathcal{H}^r_p}\}.$$

Pour $\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, on pose

$$\|\varphi\|_{\mathcal{BMO}^c} = \sup_{J\in\mathcal{D}} \Big\| \big(\frac{1}{|J|}\sum_{I\subset J} |\langle\varphi, w_I\rangle|^2\big)^{\frac{1}{2}} \Big\|_{\mathcal{M}} \tag{0.2.4}$$

et

$$\|\varphi\|_{\mathcal{BMO}^r} = \|\varphi^*\|_{\mathcal{BMO}^c(\mathbb{R},\mathcal{M})}.$$

Ce sont des normes modulo les fonctions constantes. Définissons

$$\mathcal{BMO}^c(\mathbb{R},\mathcal{M}) = \{\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^c} < \infty\},$$

$$\mathcal{BMO}^r(\mathbb{R},\mathcal{M}) = \{\varphi : \varphi^* \in \mathcal{BMO}^c(\mathbb{R},\mathcal{M})\},$$

et

$$\mathcal{BMO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \cap \mathcal{BMO}^r(\mathbb{R},\mathcal{M}).$$

Ensuite, on obtient la dualité de Fefferman désirée, et les résultats d'interpolation.

Theorem 0.2.1. *On a*

$$(\mathcal{H}_1^c(\mathbb{R},\mathcal{M}))^* = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \tag{0.2.5}$$

avec normes équivalentes.

Theorem 0.2.2. *Soit* $1 < p < \infty$*, on a*

$$[\mathcal{BMO}^c(\mathbb{R},\mathcal{M}), \mathcal{H}_1^c(\mathbb{R},\mathcal{M})]_{\frac{1}{p}} = \mathcal{H}_p^c(\mathbb{R},\mathcal{M}) \tag{0.2.6}$$

avec normes équivalentes.

Finalement, on prouve directement que notre espace $\mathcal{BMO}^c(\mathbb{R},\mathcal{M})$ est le même que dans le cadre de l'analyse harmonique à valeurs matricielles, et que nos espaces $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ sont les mêmes que ceux introduits par Mei par dualité et interpolation.

Theorem 0.2.3. *On a*

$$\mathcal{BMO}^c(\mathbb{R},\mathcal{M}) = BMO^c(\mathbb{R},\mathcal{M})$$

avec normes équivalentes. De même, $\mathcal{H}_p^c(\mathbb{R},\mathcal{M}) = H_p^c(\mathbb{R},\mathcal{M})$ *avec normes équivalentes.*

En d'autres termes, on donne une autre approche pour traiter les espaces de Hardy à valeurs opérateurs. Il doit être souligné que notre méthode est très similaire à celle utilisée dans le cas des martingales non commutatives, c'est donc beaucoup plus simple que la méthode de Mei. C'est aussi la première tentative de transférer des résultats en probabilités quantiques à l'analyse harmonique à valeurs opérateurs en utilisant l'ondelette.

0.3 Chapitre 3

Les transformées de martingales non commutatives et les opérateurs de Calderón-Zygmund rencontrés précedemment peuvent être considérés comme des opérateurs d'intégrales avec des noyaux à valeurs scalaires, de même les fonctions carrées non commutatives et les fonctions carrées à valeurs opérateurs peuvent être considérées comme des noyaux à valeurs dans un espace de Hilbert. L'objectif principal de ce chapitre est d'obtenir les estimations d'endpoint pour les OCZ ayant les noyaux qui ne commutent pas avec des fonctions, motivé par une estimation récente dans [20] pour les OCZ semi-commutatifs. Dans ce chapitre, on note $\mathcal{A} = L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{B}(\ell_2)$. Si $k(x,y)$ agit linéairement sur $\mathcal{B}(\ell_2)$ et satisfait la condition de finesse de Hörmander pour la norme des applications linéaires bornées sur $\mathcal{B}(\ell_2)$, le contenu de [20, Lemma 1.3] peut être résumé comme suit

- Si T est borné dans $L_\infty(\mathcal{B}(\ell_2); L_2^r(\mathbb{R}^n))$, alors $T : L_\infty(\mathcal{A}) \to \mathrm{BMO}_r(\mathcal{A})$,
- Si T est borné dans $L_\infty(\mathcal{B}(\ell_2); L_2^c(\mathbb{R}^n))$, alors $T : L_\infty(\mathcal{A}) \to \mathrm{BMO}_c(\mathcal{A})$.

Ici, on dit que T est borné dans $L_\infty(\mathcal{B}(\ell_2); L_2^c(\mathbb{R}^n))$ si

$$\Big\| \Big(\int_{\mathbb{R}^n} Tf(x)^* Tf(x)\, dx \Big)^{\frac{1}{2}} \Big\|_{\mathcal{B}(\ell_2)} \lesssim \Big\| \Big(\int_{\mathbb{R}^n} f(x)^* f(x)\, dx \Big)^{\frac{1}{2}} \Big\|_{\mathcal{B}(\ell_2)},$$

et $\mathrm{BMO}_c(\mathcal{A})$ désigne la version dyadique de $BMO^c(\mathbb{R}^n, \mathcal{B}(\ell_2))$ définie dans le chapitre 2 avec la norme donnée par

$$\sup_{Q \text{ cube dyadique}} \Big\| \Big(\fint_Q \big(g(x) - g_Q\big)^* \big(g(x) - g_Q\big)\, dx \Big)^{\frac{1}{2}} \Big\|_{\mathcal{B}(\ell_2)}.$$

En prenant les ajoints—afin que le $*$ passe partout de gauche à droite— on obtient la définition de T borné dans $L_\infty(L_2^r)$ et de la norme de BMO ligne. Ainsi, les arguments standard d'interpolation et de dualité démontrent que T est borné dans $L_p(\mathcal{A})$ pour $1 < p < \infty$ à condition que le noyau soit assez lisse par rapport aux deux variables et que T soit une application normale auto-adjointe bornée dans $L_\infty(L_2^r)$ et $L_\infty(L_2^c)$. En d'autre termes, les conditions de bornitude ligne/colonne jouent essentiellement le rôle de l'hypothèse de bornitude dans L_2 dans la théorie classique de Calderón-Zygmund.

Bien que cela fontionne certainement pour des noyaux non-scalaires —les actions de produit de Schur ont été utilisées dans [20, Theorem B] par exemple— les hypothèses de bornitude imposent les conditions que les noyaux commutent presque avec les fonctions, qui sont trop fortes pour les OCZ associés aux noyaux qui ne commutent pas avec les fonctions. C'est-à-dire, étant donnés $k : \mathbb{R}^{2n} \setminus \Delta \to \mathcal{B}(\ell_2)$ lisse et $x \notin \mathrm{supp}_{\mathbb{R}^n} f$, on définit formellement les OCZ ligne/colonne ainsi :

$$T_c f(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\, dy \quad \text{et} \quad T_r f(x) = \int_{\mathbb{R}^n} f(y) k(x,y)\, dy.$$

Ce n'est pas difficile de constuire des noyaux qui ne commutent pas avec les fonctions mais vérifiant

i) T_r et T_c sont bornés dans $L_2(\mathcal{A})$,

ii) T_r et T_c ne sont pas bornés dans $L_p(\mathcal{A})$ pour $1 < p \neq 2 < \infty$,

voir par exemple [43, Section 6.1] pour des exemples spécifiques. Par conséquent, les hypothèses de bornitude dans $L_\infty(L_2^r)$ et $L_\infty(L_2^c)$ sont en général trop restrictives quand le noyau et la fonction ne commutent pas. On suppose pour ce qui suit que T_r et T_c sont bornés dans $L_2(\mathcal{A})$. On s'intéresse aux formes affaiblies de bornitude dans L_p et aux estimations d'endpoint pour ces OCZ. Un OCZ *dyadique non commutant* sera une paire (T_r, T_c) qui est bornée dans $L_2(\mathcal{A})$ associée à un noyau ne commutant pas avec la fonction mais vérifiant une des conditions suivantes :

a) Noyaux dyadiques Parfaits

$$\big\| k(x,y) - k(z,y) \big\|_{\mathcal{B}(\ell_2)} + \big\| k(y,x) - k(y,z) \big\|_{\mathcal{B}(\ell_2)} = 0$$

pour tous $x, z \in Q$, $y \in R$ et Q, R des cubes dyadiques disjoints.

b) Opérateurs de shifts de Haar annulateurs

$$k(x,y) = \sum_{Q \text{ dyadic}} \sum_{\substack{R,S \text{ dyadic} \subset Q \\ \ell(R) = 2^{-r}\ell(Q) \\ \ell(S) = 2^{-s}\ell(S)}} \alpha_{RS}^Q h_R(x) h_S(y),$$

avec $r, s \in \mathbb{Z}_+$ fixés, où $\alpha_{RS}^Q \in \mathcal{B}(\ell_2)$ avec $\|\alpha_{RS}^Q\|_{\mathcal{B}(\ell_2)} \leq \frac{\sqrt{|R||S|}}{|Q|}$. Ici les h_Q sont les $2^n - 1$ fonctions de Haar relativement au cube Q.

On écrit OCZ *générique non commutant* pour des paires (T_r, T_c) qui sont bornées dans $L_2(\mathcal{A})$ et dont les noyaux ne commutent pas avec les fonctions et vérifiant les conditions de finesse standard. Notre premier résultat est le suivant.

Theorem 0.3.1. *Les inégalités suivantes sont vraies:*

i) OCZ Dyadique non commutant. *Si* $f \in L_1(\mathcal{A})$

$$\inf_{f=f_r+f_c} \big\|T_r f_r\big\|_{1,\infty} + \big\|T_c f_c\big\|_{1,\infty} \lesssim \|f\|_1.$$

ii) OCZ Générique non commutant. *Si* $f \in \mathrm{H}_1(\mathcal{A})$

$$\inf_{f=f_r+f_c} \big\|T_r f_r\big\|_1 + \big\|T_c f_c\big\|_1 \lesssim \|f\|_{\mathrm{H}_1(\mathcal{A})}.$$

$\mathrm{H}_1(\mathcal{A})$ *est la version dyadique de* $\mathcal{H}_1(\mathbb{R}^n, \mathcal{B}(\ell_2))$ *défini dans chapitre 2.*

Pour la preuve de i), on utilise la décomposition de Calderón-Zygmund non commutative et la troncature triangulaire. Pour les estimations de type ii), on se sert de la décomposition atomique et l'inégalité de John-Nirenberg dans le section 1.1. En combinant avec $L_\infty \to \mathcal{BMO}$, on obtient certaines estimations ligne/colonne L_p.

Theorem 0.3.2. *Les inégalités suivantes sont vraies pour les OCZ générique non commutant:*

i) *Si* $1 < p < 2$ *et* $f \in L_p(\mathcal{A})$

$$\inf_{f=f_r+f_c} \big\|T_r f_r\big\|_p + \big\|T_c f_c\big\|_p \lesssim \|f\|_p.$$

ii) *Si* $2 < p < \infty$ *et* $f \in L_p(\mathcal{A})$

$$\big\|T_r f\big\|_{\mathrm{H}_p^r(\mathcal{A})} + \big\|T_c f\big\|_{\mathrm{H}_p^c(\mathcal{A})} \lesssim \|f\|_p.$$

iii) *Si* $f \in L_\infty(\mathcal{A})$*, on a aussi* $\|T_r f\|_{\mathrm{BMO}_r(\mathcal{A})} + \|T_c f\|_{\mathrm{BMO}_c(\mathcal{A})} \lesssim \|f\|_\infty$.

Notre approche s'applique aussi aux paraproduits et aux tranformées de martingales dont les symboles et coefficients qui ne commutent pas avec les fonctions.

a) Transformées de martingales

$$M_\xi^r f = \sum_{k\geq 1} \Delta_k(f)\xi_{k-1} \quad \text{et} \quad M_\xi^c f = \sum_{k\geq 1} \xi_{k-1}\Delta_k(f).$$

b) Paraproduits de martingales

$$\Pi_\rho^r(f) = \sum_{k\geq 1} \mathsf{E}_{k-1}(f)\Delta_k(\rho) \quad \text{et} \quad \Pi_\rho^c(f) = \sum_{k\geq 1} \Delta_k(\rho)\mathsf{E}_{k-1}(f).$$

Ici Δ_k dénote l'opérateur de la différence de martingale $\mathsf{E}_k - \mathsf{E}_{k-1}$ et $\xi_k \in \mathcal{A}_k$ est une séquence adaptée. Bien sûr, les symboles ξ et ρ ne commutent pas nécessairement avec les fonctions.

Theorem 0.3.3. *Considérons les paires:*

i) Transformées de martingales (M_ξ^r, M_ξ^c)*, avec* $\sup_k \|\xi_k\|_{\mathcal{M}} < \infty$.

ii) Paraproduits de Martingales (Π_ρ^r, Π_ρ^c), *avec* $\Pi_\rho^{r/c}$ $L_2(\mathcal{A})$*-borné.*
Si $\Sigma_{\mathcal{A}}$ *est régulière, on obtient les inégalités de type faible* $(1,1)$ *comme dans le Théorème* 0.3.1i) *pour les tranformées et paraproduits de martingales. Les estimations dans les Théorèmes* 0.3.1ii) *et* 0.3.2 *sont aussi vraies pour les deux familles et les* $\Sigma_{\mathcal{A}}$ *arbitraires. De plus, les paraproduits de martingales* Π_ρ^r *et* Π_ρ^c *sont bornés dans* L_p *pour* $2 < p < \infty$ *et* $L_\infty \to \mathrm{BMO}$.

Nos résultats recouvrent ceux obtenus dans [51, 53] et sont dans un certain sens nets, en fournissant les remplaçants appropriés pour les coefficients ne commutant pas avec les fonctions. Notre résultat pour les paraproduits va au-delà de [33, Theorem 1.2] pour deux raisons. Premièrement, nos estimations pour $p > 2$ sont vraies pour les martingales arbitraires, pas seulement pour celles qui sont régulières. Deuxièmement, nous donnons une réponse partielle à la question de Mei dans [33] après la preuve du Théorème 1.2 pour les cas $p < 2$ et aussi pour les estimations de type faible $(1,1)$.

Introduction

The theory of quantum probability and noncommutative harmonic analysis arise from the setting of von Neumann algebras. The theory of von Neumann algebras is the natural framework for non commutative integration theory, where functions in the classical integration theory are replaced by operators on a Hilbert space, measures by traces. Historically, it is in order to study quantum mechanic that von Neumann and his collaborators laid the foundation of noncommutative integration theory. Therefore, without surprise, the study of these topics in noncommutative analysis has been partly motivated by quantum mechanics. However, the theory of quantum probability and noncommutative harmonic analysis have become independent fields of mathematical research.

The theory of noncommutative martingale inequalities is an important direction of quantum probability. The modern period of development of noncommutative martingale inequalities began with Pisier and Xu's seminal paper [48] in which the noncommutative Burkholder-Gundy inequalities and Fefferman duality theorem were established. Since then, many classical results have been successfully transferred to the noncommutative world. It should be pointed out that extending classical results to the noncommutative setting often requires additional insights. For instance, stopping time arguments and pointwise maximal function, which are often used in the classical proofs, appear unavailable in this setting. Therefore, extra techniques or theories, e.g. operator space theory, are exploited to deal with noncommutative martingales. Moreover, these techniques may yield new results even in the classical theory, as illustrated in [25].

It is well known that there are numerous interactions between classical probability theory and harmonic analysis. This interplay continues to be fruitful and play an important role in the noncommutative setting. So the second subject of this thesis deals with applications of noncommutative martingale theory to noncommutative harmonic analysis. Motivated by noncommutative martingale inequalities, Mei [34] gave a systematic study of Hardy spaces of functions with values in noncommutative L_p-spaces. His noncommutative Hardy spaces are defined by the noncommutative Lusin integral function, which are closely related to the ones defined through Poisson semigroup. Then by this connection, combined with the recently established Markov dilation property of semigroups of operators and noncommutative continuous times martingale, Junge and Mei in [19] and [35] built the theory of Hardy spaces associated with semigroups of operators, and find some important applications. Some of the applications are new even in the classical setting.

In the theory of noncommutative martingales, there is an important technique: Cuculescu's construction, which is the analogue of stopping time argument, but not strong enough as that in the classical case. However, Randrianantoanina in [51] [52] and [53] made use of this construction to prove weak $(1,1)$ type estimates of martingale transforms, (conditional) square functions. As applications, he obtained the optimal order of the best constants in several noncommutative martingale inequalities. Cuculescu's construction is also a key tool in Parcet-Randrianantoanina's noncommutative Gundy decomposition [44].

It is well known that the martingale transform and square function correspond to Hilbert transform and Littlewood-Paley function respectively in harmonic analysis. Hence Parcet [43] built the noncommutative Calderón-Zygmund decomposition based on Cuculescu's construction and obtained the weak $(1,1)$ type estimate for Calderón-Zygmund operators. Later the weak type estimates for operator-valued square functions was also obtained by Parcet and Mei [37]. A new phenomenon found by Parcet is a pseudo-localization principle which is pursed further by Hytönen [13] in the classical case.

This thesis consists of three chapters. The first chapter presents a joint work with Mei entitled "John-Nirenberg inequality and atomic decomposition for noncommutative martingales", which can be viewed as a part of noncommutative martingale theory. This work has been accepted by *J. Funct. Anal.* The content of the second chapter is on theory of Hardy spaces relating to the subject mentioned in the previous second paragraph. This chapter is a joint work with Yin entitled "Wavelet approach to operator-valued Hardy spaces", which has been accepted by *Revista Mat. Iberoa.* The last chapter is centered on Calderón-Zygmund operator with matrix-valued kernels or noncommuting martingale transforms, which is a joint work with López, Martell and Parcet entitled "Calderón-Zygmund operators associated to matrix-valued kernels".

Before I describe each chapter at length in the rest of this introduction, let us recall the main object of the three chapters, that is the noncommutative L_p spaces. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal and faithful trace τ and $S_{\mathcal{M}}^+$ be the set of all positive element x in $\mathcal{M}$ with $\tau(s(x)) < \infty$, where $s(x)$ is the smallest projection e such that $exe = x$. Let $S_{\mathcal{M}}$ be the linear span of $S_{\mathcal{M}}^+$. Then any $x \in S_{\mathcal{M}}$ has finite trace, and $S_{\mathcal{M}}$ is a w^*-dense $*$-subalgebra of $\mathcal{M}$. Let $1 \leq p < \infty$. For any $x \in S_{\mathcal{M}}$, the operator $|x|^p$ belongs to $S_{\mathcal{M}}^+$ ($|x| = (x^*x)^{\frac{1}{2}}$). We define

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}, \qquad \forall x \in S_{\mathcal{M}}.$$

One can check that $\|\cdot\|_p$ is a norm on $S_{\mathcal{M}}$. The completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ is denoted by $L_p(\mathcal{M})$ which is the usual noncommutative L_p- space associated with $(\mathcal{M}, \tau)$. For convenience, we usually set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$. The elements of $L_p(\mathcal{M})$ can be described as closed densely defined operators on H (H being the Hilbert space on which $\mathcal{M}$ acts).

0.1 Chapter 1

The theory of noncommutative martingales inequalities has been well developed in recent years. We refer, for instance, to [48], [16], [25], [27], [51] for noncommutative martingales inequalities, to [40], [2] for interpolation of noncommutative Hardy spaces and to [44], [45] for the noncommutative Gundy and Davis decompositions. There are two other works that motivate the content of chapter 1. The first one is Junge and Musat's noncommutative John-Nirenberg theorem [23] and the second the 2-atomic decomposition of the Hardy spaces $\mathcal{H}_1$ by Bekjan, Chen, Perrin and Yin [2].

Let us recall some basic notions on noncommutative martingales. In this chapter, $\mathcal{M}$ is a finite von Neumann algebra equipped with a normalized trace. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of the $\mathcal{M}_n$'s is w^*-dense in $\mathcal{M}$. Let $\mathcal{E}_n$ be the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$. A sequence $x = (x_n)_{n\geq 1}$ in $L_1(\mathcal{M})$ is called a noncommutative martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$ if $\mathcal{E}_n(x_{n+1}) = x_n$ for every $n \geq 1$. If in addition, all the x_n's are in $L_p(\mathcal{M})$ for some $1 \leq p \leq \infty$, x is called an L_p-martingale. In this case we set

$$\|x\|_p = \sup_{n\geq 1} \|x_n\|_p.$$

If $\|x\|_p < \infty$, x is called a bounded L_p-martingale. Let $x = (x_n)_{n\geq 1}$ be a noncommutative martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$. Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the convention that $x_0 = 0$ and $\mathcal{E}_0 = \mathcal{E}_1$. The sequence $dx = (dx_n)_n$ is called the martingale difference sequence of x. In the sequel, for any operator $x \in L_1(\mathcal{M})$ we denote $x_n = \mathcal{E}_n(x)$ for $n \geq 1$. For $1 \leq p < \infty$. Define $\mathcal{H}_p^c$ (resp. $\mathcal{H}_p^r$) as the completion of all finite L_p-martingales under the norm $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}_p^r} = \|S_r(x)\|_p$), where $S_c(x)$ and $S_r(x)$ are defined as

$$S_c(x) = \big(\sum_{k\geq 1} |dx_k|^2\big)^{1/2}, \quad S_r(x) = S_c(x^*).$$

The noncommutative martingale Hardy spaces $\mathcal{H}_p(\mathcal{M})$ are defined as follows: if $1 \leq p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \inf_{x=y+z} \{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\}.$$

When $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \max\{\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}\}.$$

The space $\mathcal{BMO}^c$ is defined as

$$\mathcal{BMO}^c(\mathcal{M}) = \{x \in L_1(\mathcal{M}) : \|x\|_{\mathcal{BMO}^c} < \infty\}$$

where

$$\|x\|_{\mathcal{BMO}^c} = \sup_{n\geq 1} \|\mathcal{E}_n|x - x_{n-1}|^2\|_\infty^{1/2},$$

and

$$\mathcal{BMO}^r(\mathcal{M}) = \{x : x^* \in \mathcal{BMO}^c(\mathcal{M})\}.$$

Define

$$\mathcal{BMO}(\mathcal{M}) = \mathcal{BMO}^c(\mathcal{M}) \cap \mathcal{BMO}^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{BMO}} = \max\{\|x\|_{\mathcal{BMO}^c}, \|x\|_{\mathcal{BMO}^r}\}.$$

We will also work on the conditional version of Hardy and BMO spaces developed in [25]. Let $x = (x_n)_{n\geq 1}$ be a finite martingale in $L_2(\mathcal{M})$. We set

$$s_c(x) = \big(\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k|^2\big)^{1/2} \quad \text{and} \quad s_r(x) = s_c(x^*).$$

Let $0 < p < \infty$. Define $\mathsf{h}_p^c(\mathcal{M})$ (resp. $\mathsf{h}_p^r(\mathcal{M})$) as the completion of all finite L_∞-martingales under the (quasi-)norm $\|x\|_{\mathsf{h}_p^c} = \|s_c(x)\|_p$ (resp. $\|x\|_{\mathsf{h}_p^r} = \|s_r(x)\|_p$). Define $\mathsf{h}_p^d(\mathcal{M})$ as the subspace of $\ell_p(L_p(\mathcal{M}))$ consisting of all martingale difference sequences, where $\ell_p(L_p(\mathcal{M}))$ is the space of all sequences $a = (a_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ such that

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \big(\sum_{n\geq 1} \|a_n\|_p^p\big)^{1/p} < \infty$$

with the usual modification for $p = \infty$. The noncommutative conditional martingale Hardy spaces are defined as follows: if $0 < p < 2$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M}) + \mathsf{h}_p^d(\mathcal{M})$$

equipped with the (quasi-)norm

$$\|x\|_{\mathsf{h}_p} = \inf_{x=y+z+w}\{\|y\|_{\mathsf{h}_p^c} + \|z\|_{\mathsf{h}_p^r} + \|w\|_{\mathsf{h}_p^d}\}.$$

When $2 \leq p < \infty$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M}) \cap \mathsf{h}_p^d(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathsf{h}_p} = \max\{\|x\|_{\mathsf{h}_p^c}, \|x\|_{\mathsf{h}_p^r}, \|x\|_{\mathsf{h}_p^d}\}.$$

The space bmo^c is defined as

$$\mathsf{bmo}^c(\mathcal{M}) = \{x \in L_1(\mathcal{M}) : \|x\|_{\mathsf{bmo}^c} < \infty\}$$

where

$$\|x\|_{\mathsf{bmo}^c} = \max\left\{\|\mathcal{E}_1(x)\|_\infty, \quad \sup_{n\geq 1}\|\mathcal{E}_n|x - x_n|^2\|_\infty^{1/2}\right\}.$$

Let

$$\mathsf{bmo}^r(\mathcal{M}) = \{x : x^* \in \mathsf{bmo}^c(\mathcal{M})\}.$$

Let $\mathsf{bmo}^d(\mathcal{M})$ be the subspace of $\ell_\infty(L_\infty(\mathcal{M}))$ consisting of all martingale difference sequences. Note that $\mathsf{bmo}^d(\mathcal{M}) = \mathsf{h}_\infty^d(\mathcal{M})$. Define

$$\mathsf{bmo}(\mathcal{M}) = \mathsf{bmo}^c(\mathcal{M}) \cap \mathsf{bmo}^r(\mathcal{M}) \cap \mathsf{bmo}^d(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathsf{bmo}} = \max\{\|x\|_{\mathsf{bmo}^c}, \|x\|_{\mathsf{bmo}^r}, \|x\|_{\mathsf{bmo}^d}\}.$$

0.1.1 John-Nirenberg inequality

We begin with recalling the classical John-Nirenberg inequalities in the martingale theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n\geq 0}$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$ with the associated conditional expectations $(\mathbb{E}_n)_{n\geq 0}$. The $BMO(\Omega)$ space is defined as the set of all $x \in L_1(\Omega)$ with the norm

$$\|x\|_{BMO} = \sup_n \|\mathbb{E}_n|x - x_{n-1}|\|_\infty < \infty.$$

The classical John-Nirenberg theorem says that there exist two universal constants c_1, $c_2 > 0$ such that if $\|x\|_{BMO} < c_2$, then

$$\sup_n \|\mathbb{E}_n(e^{c_1|x-x_{n-1}|})\|_\infty < 1, \tag{0.1.1}$$

which is equivalent to the statement: For any $n \geq 1$, $E \in \mathcal{F}_n$ and $\lambda > 0$, there exists a universal constant $c > 0$ such that

$$\frac{1}{\mathbb{P}(E)}\,\mathbb{P}\big(\{\omega \in E \,:\, |x(\omega) - x_{n-1}(\omega)| > \lambda\}\big) \leq c_2 \exp(-c\lambda/\|x\|_{BMO}). \tag{0.1.2}$$

There remains a equivalent characterization: There exists an absolute constant c such that for all $1 \leq p < \infty$,

$$\|x\|_{BMO} \leq \sup_n \sup_{E\in\mathcal{F}_n} \frac{1}{\mathbb{P}(E)^{1/p}} \|(x - x_{n-1})\mathbb{1}_E\|_p \leq cp\|x\|_{BMO}. \tag{0.1.3}$$

Junge and Musat [23] proved a noncommutative version of John-Nirenberg theorem resembling (0.1.3): There exists an absolute constant c such that for all $2 \leq p < \infty$,

$$\|x\|_{\mathcal{BMO}} \leq \mathcal{B}_p(x) \leq cp\|x\|_{\mathcal{BMO}},$$

where

$$\begin{aligned}\mathcal{B}_p(x) = \max\{ &\sup_n \sup_{b\in\mathcal{M}_n,\|b\|_p\leq 1} \|(x - x_{n-1})b\|_p,\\ &\sup_n \sup_{b\in\mathcal{M}_n,\|b\|_p\leq 1,} \|b(x - x_{n-1})\|_p\}.\end{aligned}$$

However, this theorem does not hold (see Remark 2.14 for a counterexample) when considering $\mathcal{BMO}^c(\mathcal{M})$ or $\mathcal{BMO}^r(\mathcal{M})$ separately. On the other hand, it does not correspond to the commonly used form of the classical John-Nirenberg inequality. The first purpose of this chapter is to remedy these aspects of Junge and Musat's theorem. The following is one of our main results. In this chapter, $\mathcal{P}(\mathcal{M})$ denotes the set of all projections of $\mathcal{M}$.

Theorem 0.1.1. *For all $0 < p < \infty$,*

$$\alpha_p^{-1}\|x\|_{\mathsf{bmo}^c} \leq \|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} \leq \beta_p\|x\|_{\mathsf{bmo}^c},$$

where

$$\|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} = \max\Big\{\|\mathcal{E}_1(x)\|_\infty, \quad \sup_n \sup_{e\in\mathcal{P}(\mathcal{M}_n)} \frac{1}{(\tau(e))^{1/p}}\|(x - x_n)e\|_{\mathsf{h}^c_p}\Big\}.$$

The two constants α_p and β_p have the following properties

(i) $\alpha_p = 1$ *for* $2 \leq p < \infty$*;*

(ii) $\alpha_p \leq C^{1/p-1/2}$ *for* $0 < p < 2$*;*

(iii) $\beta_p \leq cp$ *for* $2 \leq p < \infty$*;*

(iv) $\beta_p = 1$ *for* $0 < p < 2$*.*

A similar result holds for $\mathcal{BMO}^c(\mathcal{M})$ but only with $2 \leq p < \infty$ (see Remark 2.9). On the other hand, our proof of Theorem D can be easily modified to extend Junge/Musat's result to all $0 < p < \infty$ and in the form (0.1.3) (see Corollary 2.19). Moreover the optimal order cp obtained using very recent results of Randrianantoanina [52] enable us to formulate the inequality in the form (0.1.1) and (0.1.2).

In the last section of chapter 1, we give an negative answer to an open question asked in [23] (on page 136) that given $2 < p < \infty$, whether there exists a constant c_p such that

$$\sup_k \|\mathcal{E}_k|x - \mathcal{E}_{k-1}x|^p\|_\infty^{\frac{1}{p}} \leq c_p\|x\|_{\mathcal{BMO}}?$$

Theorem 0.1.2. *Suppose* $\sup_k \|\mathcal{E}_k|f-\mathcal{E}_{k-1}f|^p\|_\infty^{1/p} \leq c_p(n)\|f\|_{\mathcal{BMO}}$ *for some* $p\geq 3$. *Then*

$$c_p(n) \geq c(\log(n+1))^{\frac{2}{p}}.$$

0.1.2 Atomic decomposition

We now turn to the second objective of this chapter: the atomic decomposition of different noncommutative Hardy spaces. Let us recall the 2-atomic decomposition obtained in [2]. An element $a \in L_1(\mathcal{M})$ is said to be a $(1,2)_c$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and $e \in \mathcal{P}(\mathcal{M}_n)$ such that

(i) $\mathcal{E}_n(a) = 0$; (ii)$ae = a$; (iii) $\|a\|_2 \leq (\tau(e))^{-1/2}$.

The atomic Hardy space $\mathsf{h}_{1,\mathrm{at}}^c(\mathcal{M})$ is defined as the space of all $x \in L_1(\mathcal{M})$, such that the following $\|\cdot\|_{\mathsf{h}_{1,\mathrm{at}}^c}$ norm is finite,

$$\|x\|_{\mathsf{h}_{1,\mathrm{at}}^c} = \|\mathcal{E}_1 x\|_1 + \inf \sum_j |\lambda_j|.$$

Here the infimum is taken for possible decompositions $x - \mathcal{E}_1 x = \sum_j \lambda_j a_j$ with $\lambda_j \in \mathbb{C}$, a_j being $(1,2)_c$-atom. It is proved in [2] that $x \in \mathsf{h}_1^c(\mathcal{M})$ if and only if $x \in \mathsf{h}_{1,\mathrm{at}}^c(\mathcal{M})$ and

$$\|x\|_{\mathsf{h}_1^c} \simeq \|x\|_{\mathsf{h}_{1,\mathrm{at}}^c}.$$

Together with the equivalence $\mathcal{H}_1^c(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M}) + \mathsf{h}_1^d(\mathcal{M})$, the authors of [2] also obtained a 2-atomic decomposition for $\mathcal{H}_1^c(\mathcal{M})$.

Let us briefly recall the argument used in [2]. The dual space of $\mathrm{h}_{1,\mathrm{at}}^c(\mathcal{M})$ can be described as

$$\Lambda^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\Lambda^c} < \infty\}$$

with

$$\|x\|_{\Lambda^c} = \max\{\|\mathcal{E}_1 x\|_\infty, \quad \sup_{n\geq 1}\sup_{e\in\mathcal{P}_n} (\frac{1}{\tau(e)}\tau(e|x-x_n|^2))^{\frac{1}{2}}\}.$$

Actually, the supremum in the definition above can be taken for all $b \in L_1(\mathcal{M}_n)$ since the extreme points of the unit ball of $L_1(\mathcal{M}_n)$ are all multiples of projections. Therefore,

$$\begin{aligned} \|x\|_{\Lambda^c} &= \max\{\|\mathcal{E}_1 x\|_\infty, \quad \sup_{n\geq 1}\sup_{b\in\mathcal{M}_n} (\frac{1}{\|b\|_1}\tau(b|x-x_n|^2))^{\frac{1}{2}}\} \qquad (0.1.4)\\ &= \max\{\|\mathcal{E}_1 x\|_\infty, \quad \sup_{n\geq 1}\|\mathcal{E}_n|x-x_n|^2\|_\infty^{\frac{1}{2}}\}\\ &= \|x\|_{\mathsf{bmo}^c}. \end{aligned}$$

Then the duality $\mathsf{h}_1^c(\mathcal{M}) = \mathsf{bmo}^c(\mathcal{M})$ yields $\mathsf{h}_{1,\mathrm{at}}^c(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M})$.

It is well known in the classical theory that 2-atoms in the previous atomic decomposition can be replaced by q-atoms for any $1 < q \leq \infty$. Let us recall these atoms in the commutative case. A function $a \in L_1(\Omega)$ is said to be a q-atom if there exist $n \geq 1$ and $E \in \mathcal{F}_n$ such that

(i) $\mathbb{E}_n a = 0$; (ii) $\{a \neq 0\} \subset E$; (iii) $\|a\|_q \leq \mathbb{P}(E)^{-1+\frac{1}{q}}$.

We refer to [57] for more information.

The main difficulty to obtain q-atomic decompositions in the noncommutative case is that the key equivalence (1.0.8) no longer holds if one replaces the power indices 2 by $q' \neq 2$, $1 \leq q' < \infty$. We overcome this obstacle by Theorem D.

Theorem 0.1.3. *For all* $1 < q \leq \infty$,

$$\mathsf{h}_1^c(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at_{q,pr}}}^c(\mathcal{M})$$

with equivalent norms. Here $\mathrm{h}^{\mathrm{c}}_{1,\mathrm{at_{q,pr}}}(\mathcal{M})$ *is the* q*-atomic Hardy spaces with its atoms defined as:* a *is called a* q*-atom if there exist* $n \geq 1$ *and a projection* $e \in \mathcal{P}(\mathcal{M}_n)$ *such that*

(i)*)* $\mathcal{E}_n(a) = 0$*;* (ii) $ae = a$*;* (iii) $\|a\|_{\mathsf{h}_q^c} \leq (\tau(e))^{-\frac{1}{q'}}$.

Note that $\mathsf{h}^c_{1,\mathrm{at_{2,pr}}}(\mathcal{M}) = \mathsf{h}^c_{1,\mathrm{at}}(\mathcal{M})$, so we recover the 2-atomic decomposition of [2]. Moreover, applying the conditional version of Junge/Musat's theorem in the form (0.1.3), we get a q-atomic decomposition for $\mathsf{h}_1(\mathcal{M})$ in which the atoms are defined in a similar way with $\| \, \|_{\mathsf{h}_q^c}$ in (iii) above replaced by $\| \, \|_q$ and the support condition (ii) weakened to $r(a) \leq e$ or $l(a) \leq e$ (see Theorem 3.19). This is exactly the noncommutative analogue of the classical atomic decomposition.

The John-Nirenberg inequality and atomic decomposition established here will be applied to prove the $H_1 \to L_1$ type estimates in Chapter 3.

0.2 Chapter 2

Motivated by quantum probability mentioned in chapter 1 and matrix-valued harmonic analysis, Mei in [34] defined H_p spaces for operator-valued functions by considering the operator-valued Lusin's square function. For $1 \leq p < \infty$, his $H_p^c(\mathbb{R}, \mathcal{M})$ is defined to be the completion of the space of all $S_{\mathcal{M}}$-valued simple functions f's with norm $\|S_c(f)\|_p$ finite, where $S_c(f)$ is the noncommutative analogue of the classical Lusin integral defined by

$$S_c(f)(x) = \Big(\int_\Gamma \Big[\frac{\partial f^*}{\partial t} \frac{\partial f}{\partial t} + \sum\nolimits_j \frac{\partial f^*}{\partial y_j} \frac{\partial f}{\partial y_j} \Big] (x+y,t) \, \frac{dydt}{t^{n-1}} \Big)^{\frac{1}{2}}$$

with $\Gamma = \{(y,t) \in \mathbb{R}_+^{n+1} \mid |y| < t\}$ and $f(y,t) = P_t f(y)$ for the Poisson semigroup $(P_t)_{t \geq 0}$. Then $H_p^c(\mathbb{R}, \mathcal{M})$ and $H_p(\mathbb{R}, \mathcal{M})$ are defined in the same way as those in martingale case.

The remarkable result proved by Mei is that $H_1^c(\mathbb{R}, \mathcal{M})$ is the predual of the BMO space appeared in matrix-valued harmonic analysis denoted by $BMO^c(\mathbb{R}, \mathcal{M})$. This space is defined to be the space of all $\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, dt/1+t^2))$ such that

$$\|\varphi\|_{BMO^c} = \sup_{I \subset \mathbb{R}} \Big\| \frac{1}{|I|} \int_I |\varphi - \varphi_I|^2 \Big\|_{\mathcal{M}}^{\frac{1}{2}} < \infty.$$

He also obtained desired interpolation results and Littewood-Paley inequality. Mei adapted the classical approach in his study and reduced many problems to the martingale case. To do so he invented a very interesting and powerful technique: the BMO space on $\mathbb{R}$ (both in the classical and noncommutative cases) is the intersection of two dyadic BMO spaces.

On the other hand, in classical harmonic analysis, it is well known that $H_1(\mathbb{R})$ defined by square function and $\mathcal{H}_1(\mathbb{R})$ defined by wavelets coincide since they admit the same atomic decomposition. As explained in chapter 1, it is very difficult to obtain the atomic decomposition for noncommutative Hardy spaces by explicit construction as in the classical situation, hence we cannot compare these two spaces by the classical way. In this chapter, we directly define $\mathcal{H}_p(\mathbb{R}, \mathcal{M})$ and $\mathcal{BMO}(\mathbb{R}, \mathcal{M})$ via wavelets. The definitions are similar to the ones in the martingale case, but associated to a fixed wavelet basis of $(w_I)_{I \in \mathcal{D}}$. In this chapter, for simplicity, we denote $L_\infty(\mathbb{R})\bar{\otimes}\mathcal{M}$ by $\mathcal{N}$. For $f \in S_{\mathcal{N}}$, the square functions are defined as

$$S_c(f)(x) = \Big(\sum_{I\in\mathcal{D}} \frac{|\langle f, w_I\rangle|^2}{|I|} \mathbb{1}_I(x)\Big)^{\frac{1}{2}}. \tag{0.2.1}$$

and $S_r(f) = S_c(f^*)$. The norms are defined as

$$\|f\|_{\mathcal{H}_p^c} = \|S_c(f)\|_{L_p(\mathcal{N})}, \qquad \text{and} \quad \|f\|_{\mathcal{H}_p^r} = \|S_r(f)\|_{L_p(\mathcal{N})}.$$

So the spaces $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathbb{R},\mathcal{M})$) are defined to the space of the completion of $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})})$ (resp. $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})})$. We then define the operator-valued Hardy spaces as follows: for $1 \le p < 2$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M}) = \mathcal{H}_p^c(\mathbb{R},\mathcal{M}) + \mathcal{H}_p^r(\mathbb{R},\mathcal{M}) \tag{0.2.2}$$

with the norm

$$\|f\|_{\mathcal{H}_p} = \inf\{\|g\|_{\mathcal{H}_p^c} + \|h\|_{\mathcal{H}_p^r} : f = g + h, g \in \mathcal{H}_p^c, h \in \mathcal{H}_p^r\}$$

and for $2 \le p < \infty$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M}) = \mathcal{H}_p^c(\mathbb{R},\mathcal{M}) \cap \mathcal{H}_p^r(\mathbb{R},\mathcal{M}) \tag{0.2.3}$$

with the norm defined as

$$\|f\|_{\mathcal{H}_p} = \max\{\|f\|_{\mathcal{H}_p^c}, \|f\|_{\mathcal{H}_p^r}\}.$$

For $\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, set

$$\|\varphi\|_{\mathcal{BMO}^c} = \sup_{J\in\mathcal{D}} \Big\|\big(\frac{1}{|J|}\sum_{I\subset J} |\langle \varphi, w_I\rangle|^2\big)^{\frac{1}{2}}\Big\|_{\mathcal{M}} \tag{0.2.4}$$

and

$$\|\varphi\|_{\mathcal{BMO}^r} = \|\varphi^*\|_{\mathcal{BMO}^c(\mathbb{R},\mathcal{M})}.$$

These are norms modulo constant functions. Define

$$\mathcal{BMO}^c(\mathbb{R},\mathcal{M}) = \{\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^c} < \infty\},$$

$$\mathcal{BMO}^r(\mathbb{R},\mathcal{M}) = \{\varphi : \varphi^* \in \mathcal{BMO}^c(\mathbb{R},\mathcal{M})\},$$

and

$$\mathcal{BMO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \cap \mathcal{BMO}^r(\mathbb{R},\mathcal{M}).$$

Then we obtain the desired Fefferman duality theorem, and interpolation results.

Theorem 0.2.1. *We have*

$$(\mathcal{H}_1^c(\mathbb{R},\mathcal{M}))^* = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \tag{0.2.5}$$

with equivalent norms.

Theorem 0.2.2. *Let $1 < p < \infty$, we have*

$$[\mathcal{BMO}^c(\mathbb{R},\mathcal{M}), \mathcal{H}_1^c(\mathbb{R},\mathcal{M})]_{\frac{1}{p}} = \mathcal{H}_p^c(\mathbb{R},\mathcal{M}) \tag{0.2.6}$$

with equivalent norms.

At last, we directly prove our $\mathcal{BMO}^c(\mathbb{R},\mathcal{M})$ is as same as the one in matrix-valued harmonic analysis, hence our $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ are the same as Mei's by duality and interpolation.

Theorem 0.2.3. *We have*

$$\mathcal{BMO}^c(\mathbb{R},\mathcal{M}) = BMO^c(\mathbb{R},\mathcal{M})$$

with equivalent norms. Similarly, $\mathcal{H}_p^c(\mathbb{R},\mathcal{M}) = H_p^c(\mathbb{R},\mathcal{M})$ with equivalent norms.

In other words, we give another approach to deal with operator-valued Hardy spaces. It should be pointed out that our method is very parallel to the way in noncommutative martingale case, hence it is much simpler than Mei's method. This is also the first attempt by using wavelet to transfer the results in quantum probability to operator-valued harmonic analysis.

0.3 Chapter 3

The noncommutative martingale transforms and Calderón-Zygmund operators considered before can be viewed as integral operators with scalar-valued kernels, while noncommutative square functions and operator-valued square functions Hilbert-valued kernels. The main goal of this chapter is to obtain endpoint estimates for CZO's with noncommuting kernels, motivated by a recent estimate from [20] for semicommutative CZO's. In this chapter, we denote $\mathcal{A} = L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{B}(\ell_2)$ for simplification. If $k(x,y)$ acts linearly on $\mathcal{B}(\ell_2)$ and satisfies the Hörmander smoothness condition in the norm of bounded linear maps on $\mathcal{B}(\ell_2)$, the content of [20, Lemma 1.3] can be summarized as follows

- If T is $L_\infty(\mathcal{B}(\ell_2); L_2^r(\mathbb{R}^n))$-bounded, then $T : L_\infty(\mathcal{A}) \to \mathrm{BMO}_r(\mathcal{A})$,
- If T is $L_\infty(\mathcal{B}(\ell_2); L_2^c(\mathbb{R}^n))$-bounded, then $T : L_\infty(\mathcal{A}) \to \mathrm{BMO}_c(\mathcal{A})$.

Here, the $L_\infty(L_2^c)$-boundedness assumption refers to

$$\Big\| \Big(\int_{\mathbb{R}^n} Tf(x)^* Tf(x)\, dx \Big)^{\frac{1}{2}} \Big\|_{\mathcal{B}(\ell_2)} \lesssim \Big\| \Big(\int_{\mathbb{R}^n} f(x)^* f(x)\, dx \Big)^{\frac{1}{2}} \Big\|_{\mathcal{B}(\ell_2)},$$

while $\mathrm{BMO}_c(\mathcal{A})$ is the dyadic version of $BMO^c(\mathbb{R}^n, \mathcal{B}(\ell_2))$ defined in chapter 2 with norms given by

$$\sup_{Q \text{ dyadic cube}} \Big\| \Big(\fint_Q \big(g(x) - g_Q\big)^* \big(g(x) - g_Q\big)\, dx \Big)^{\frac{1}{2}} \Big\|_{\mathcal{B}(\ell_2)}.$$

Taking adjoints —so that the $*$ switches everywhere from left to right— we find $L_\infty(L_2^r)$-boundedness and the row-BMO norm. Thus, standard interpolation and duality arguments show that $T : L_p(\mathcal{A}) \to L_p(\mathcal{A})$ for $1 < p < \infty$ provided the kernel is smooth enough in both variables and T is a normal self-adjoint map satisfying the $L_\infty(L_2^r)$ and $L_\infty(L_2^c)$ boundedness assumptions. In other words, the row/column boundedness conditions essentially play the role of the L_2-boundedness assumption in classical Calderón-Zygmund theory.

Although this certainly works for non-scalar kernels —Schur product actions were used e.g. in [20, Theorem B]— the boundedness assumptions impose nearly commuting conditions on the kernel which are too strong for CZO's associated to noncommuting kernels. Namely, given $k : \mathbb{R}^{2n} \setminus \Delta \to \mathcal{B}(\ell_2)$ smooth and given $x \notin \mathrm{supp}_{\mathbb{R}^n} f$, let us set formally the row/column CZO's

$$T_c f(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\, dy \quad \text{and} \quad T_r f(x) = \int_{\mathbb{R}^n} f(y) k(x,y)\, dy.$$

It is not difficult to construct noncommuting kernels with

i) T_r and T_c are $L_2(\mathcal{A})$-bounded,

ii) T_r and T_c are not $L_p(\mathcal{A})$-bounded for $1 < p \neq 2 < \infty$,

see e.g. [43, Section 6.1] for specific examples. Therefore, the $L_\infty(L_2^r)$ and $L_\infty(L_2^c)$ boundedness assumption is in general too restrictive when kernel and function do not commute. Assume for what follows that T_r and T_c are $L_2(\mathcal{A})$-bounded. We are interested in weakened forms of L_p boundedness and endpoint estimates for these CZO's. A *dyadic noncommuting* CZO will be a $L_2(\mathcal{A})$-bounded pair (T_r, T_c) associated to a noncommuting kernel satisfying one of the following conditions:

a) Perfect dyadic kernels

$$\big\|k(x,y) - k(z,y)\big\|_{\mathcal{B}(\ell_2)} + \big\|k(y,x) - k(y,z)\big\|_{\mathcal{B}(\ell_2)} = 0$$

whenever $x, z \in Q$ and $y \in R$ for some disjoint dyadic cubes Q, R.

b) Cancellative Haar shift operators

$$k(x,y) = \sum_{Q \text{ dyadic}} \sum_{\substack{R,S \text{ dyadic} \subset Q \\ \ell(R) = 2^{-r}\ell(Q) \\ \ell(S) = 2^{-s}\ell(S)}} \alpha_{RS}^Q h_R(x) h_S(y),$$

for some fixed $r, s \in \mathbb{Z}_+$ where the $\alpha_{RS}^Q \in \mathcal{B}(\ell_2)$ with $\|\alpha_{RS}^Q\|_{\mathcal{B}(\ell_2)} \leq \frac{\sqrt{|R||S|}}{|Q|}$. Here h_Q refers to any of the $2^n - 1$ Haar functions related to the cube Q.

We write *generic noncommuting* CZO for $L_2(\mathcal{A})$-bounded pairs (T_r, T_c) with a noncommuting kernel satisfying the standard smoothness. Our first result is the following.

Theorem 0.3.1. *The following inequalities hold*:

i) Dyadic noncommuting CZO's. *Given* $f \in L_1(\mathcal{A})$

$$\inf_{f = f_r + f_c} \big\|T_r f_r\big\|_{1,\infty} + \big\|T_c f_c\big\|_{1,\infty} \lesssim \|f\|_1.$$

ii) Generic noncommuting CZO's. *Given* $f \in \mathrm{H}_1(\mathcal{A})$

$$\inf_{f = f_r + f_c} \big\|T_r f_r\big\|_1 + \big\|T_c f_c\big\|_1 \lesssim \|f\|_{\mathrm{H}_1(\mathcal{A})}.$$

Again, $\mathrm{H}_1(\mathcal{A})$ *is the dyadic version of* $\mathcal{H}_1(\mathbb{R}^n, \mathcal{B}(\ell_2))$ *defined in chapter 2.*

For the proof of i), we use the noncommutative Calderón-Zygmund decomposition and triangular truncation. For the type estimates of ii), we use the atomic decomposition and John-Nirenberg inequality in section 1.1. In conjunction with $L_\infty \to \mathcal{BMO}$, we get certain row/column L_p estimates.

Theorem 0.3.2. *The following inequalities hold for generic noncommuting CZO's*:

i) *If* $1 < p < 2$ *and* $f \in L_p(\mathcal{A})$

$$\inf_{f = f_r + f_c} \big\|T_r f_r\big\|_p + \big\|T_c f_c\big\|_p \lesssim \|f\|_p.$$

ii) *If* $2 < p < \infty$ *and* $f \in L_p(\mathcal{A})$

$$\big\|T_r f\big\|_{\mathrm{H}_p^r(\mathcal{A})} + \big\|T_c f\big\|_{\mathrm{H}_p^c(\mathcal{A})} \lesssim \|f\|_p.$$

iii) *Given* $f \in L_\infty(\mathcal{A})$, *we also have* $\|T_r f\|_{\mathrm{BMO}_r(\mathcal{A})} + \|T_c f\|_{\mathrm{BMO}_c(\mathcal{A})} \lesssim \|f\|_\infty$.

Our approach also applies to noncommutative paraproducts and martingale transforms with noncommuting symbols/coefficients.

a) Noncommuting martingale transforms

$$M_\xi^r f = \sum_{k\geq 1} \Delta_k(f)\xi_{k-1} \quad \text{and} \quad M_\xi^c f = \sum_{k\geq 1} \xi_{k-1}\Delta_k(f).$$

b) Paraproducts with noncommuting symbol

$$\Pi_\rho^r(f) = \sum_{k\geq 1} \mathsf{E}_{k-1}(f)\Delta_k(\rho) \quad \text{and} \quad \Pi_\rho^c(f) = \sum_{k\geq 1} \Delta_k(\rho)\mathsf{E}_{k-1}(f).$$

Here Δ_k denotes the martingale difference operator $\mathsf{E}_k - \mathsf{E}_{k-1}$ and $\xi_k \in \mathcal{A}_k$ is an adapted sequence. Of course, the symbols ξ and ρ do not necessarily commute with the function.

Theorem 0.3.3. *Consider the pairs*:

i) Martingale transforms (M_ξ^r, M_ξ^c), *with* $\sup_k \|\xi_k\|_{\mathcal{M}} < \infty$.

ii) Martingale paraproducts (Π_ρ^r, Π_ρ^c), *with* $\Pi_\rho^{r/c}$ $L_2(\mathcal{A})$*-bounded.*

If $\Sigma_{\mathcal{A}}$ *is regular, we obtain weak type* $(1,1)$ *inequalities like in Theorem* 0.3.1i) *for martingale transforms and paraproducts . The estimates in Theorems* 0.3.1ii) *and* 0.3.2 *also hold for both families and for arbitrary filtrations* $\Sigma_{\mathcal{A}}$*. Moreover, the martingale paraproducts* Π_ρ^r *and* Π_ρ^c *are* L_p*-bounded for* $2 < p < \infty$ *and* $L_\infty \to \mathrm{BMO}$.

Our results recover those in [51, 53] and are in some sense sharp, providing appropriate substitutes for noncommuting coefficients. Our result for paraproducts goes beyond [33, Theorem 1.2] in two aspects. First, our estimates for $p > 2$ hold for arbitrary martingales, not just for regular ones. Second, we give a partial answer to Mei's question in [33] after the proof of Theorem 1.2 for the case $p < 2$ and also for the weak type $(1,1)$ estimates.

Chapter 1

John-Nirenberg inequality and atomic decomposition for noncommutative martingales

Introduction

This chapter deals with BMO spaces and atomic decomposition for noncommutative martingales. The modern period of development of noncommutative martingale inequalities began with Pisier and Xu's seminal paper [48] in which they established the noncommutative Burkholder-Gundy inequalities and Fefferman duality theorem between $\mathcal{H}_1$ and $\mathcal{BMO}$. Since then remarkable progress has been made in the field. We refer, for instance, to [16], [25], [27], [51] for other noncommutative martingales inequalities, to [40], [2] for interpolation of noncommutative Hardy spaces and to [44], [45] for the noncommutative Gundy and Davis decompositions. Let us also mention two other works that motivate the present chapter. The first one is Junge and Musat's noncommutative John-Nirenberg theorem [23] and the second the 2-atomic decomposition of the Hardy spaces $\mathcal{H}_1$ by Bekjan, Chen, Perrin and Yin [2].

Before describing our main results, we recall the classical John-Nirenberg inequalities in the martingale theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n\geq 0}$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$ with the associated conditional expectations $(\mathbb{E}_n)_{n\geq 0}$. The $BMO(\Omega)$ space is defined as the set of all $x \in L_1(\Omega)$ with the norm

$$\|x\|_{BMO} = \sup_n \|\mathbb{E}_n|x - x_{n-1}|\|_\infty < \infty. \tag{1.0.1}$$

The classical John-Nirenberg theorem says that there exist two universal constants c_1, $c_2 > 0$ such that if $\|x\|_{BMO} < c_2$, then

$$\sup_n \|\mathbb{E}_n(e^{c_1|x-x_{n-1}|})\|_\infty < 1. \tag{1.0.2}$$

This statement is equivalent to the following one: there exists an absolute constant c such that for all $1 \leq p < \infty$,

$$\|x\|_{BMO} \leq \sup_n \|\mathbb{E}_n|x - x_{n-1}|^p\|_\infty^{\frac{1}{p}} \leq cp\|x\|_{BMO}. \tag{1.0.3}$$

A duality argument yields

$$\|\mathbb{E}_n|x - x_{n-1}|^p\|_\infty^{\frac{1}{p}} = \sup_{b\in L_\infty(\mathcal{F}_n), \|b\|_1\leq 1} \left(\int |x - x_{n-1}|^p b d\mathbb{P}\right)^{\frac{1}{p}} \tag{1.0.4}$$

$$= \sup_{b\in L_\infty(\mathcal{F}_n),\|b\|_p\le 1} \|(x-x_{n-1})b\|_p. \tag{1.0.5}$$

Furthermore, by the extreme point property of $L_1(\mathcal{F}_n)$ and (1.0.4), the John-Nirenberg theorem (1.0.3) can be rewritten as follows

$$\|x\|_{BMO} \le \sup_n \sup_{E\in\mathcal{F}_n} \frac{1}{\mathbb{P}(E)}\|(x-x_{n-1})\mathbb{1}_E\|_p \le cp\|x\|_{BMO}. \tag{1.0.6}$$

Accordingly, (1.0.2) can be reformulated as: For any $n\ge 1$, $E\in\mathcal{F}_n$ and $\lambda>0$

$$\frac{1}{\mathbb{P}(E)}\,\mathbb{P}\big(\{\omega\in E : |x(\omega)-x_{n-1}(\omega)|>\lambda\}\big) \le c_2\exp(-c_1\lambda/\|x\|_{BMO}). \tag{1.0.7}$$

Junge and Musat [23] proved a noncommutative version of John-Nirenberg theorem corresponding to (1.0.5). To state their result we need fix some notation. Let $\mathcal{M}$ be a finite von Neumann algebra with a normal faithful tracial state τ. Let $(\mathcal{M}_n)_{n\ge1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of $\mathcal{M}_n$'s is w^*-dense in $\mathcal{M}$. Let $\mathcal{E}_n$ be the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$. Define

$$\|x\|_{\mathcal{BMO}^c} = \sup_{n\ge1} \|\mathcal{E}_n|x-x_{n-1}|^2\|_\infty^{\frac12}$$

and

$$\mathcal{BMO}(\mathcal{M}) = \{x\in L_1(\mathcal{M}) : \|x\|_{\mathcal{BMO}}<\infty\}$$

with

$$\|x\|_{\mathcal{BMO}} = \max\{\|x\|_{\mathcal{BMO}^c}, \|x^*\|_{\mathcal{BMO}^c}\}.$$

Then Junge and Musat's John-Nirenberg inequality reads as follows: there exists an absolute constant c such that for all $2\le p<\infty$,

$$\|x\|_{\mathcal{BMO}} \le \mathcal{B}_p(x) \le cp\|x\|_{\mathcal{BMO}},$$

where

$$\begin{aligned}\mathcal{B}_p(x) = \max\{ &\sup_n \sup_{b\in\mathcal{M}_n,\|b\|_p\le1} \|(x-x_{n-1})b\|_p,\\ &\sup_n \sup_{b\in\mathcal{M}_n,\|b\|_p\le1,} \|b(x-x_{n-1})\|_p\}.\end{aligned}$$

However, this theorem does not hold (see Remark 2.14 for a counterexample) when considering $\mathcal{BMO}^c(\mathcal{M})$ or $\mathcal{BMO}^r(\mathcal{M})$ separately. On the other hand, it does not correspond to the commonly used form of the classical John-Nirenberg inequality. The first purpose of this paper is to remedy these aspects of Junge and Musat's theorem. The following is one of our main results. We refer to the next section for all spaces and notation used below. $\mathcal{P}(\mathcal{M})$ denotes the set of all projections of $\mathcal{M}$.

Theorem D. *For all* $0<p<\infty$,

$$\alpha_p^{-1}\|x\|_{\mathsf{bmo}^c} \le \|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} \le \beta_p\|x\|_{\mathsf{bmo}^c},$$

where

$$\|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} = \max\big\{\|\mathcal{E}_1(x)\|_\infty,\quad \sup_n \sup_{e\in\mathcal{P}(\mathcal{M}_n)} \frac{1}{(\tau(e))^{1/p}}\|(x-x_n)e\|_{\mathsf{h}^c_p}\big\}.$$

The two constants α_p and β_p have the following properties

(i) $\alpha_p = 1$ *for* $2 \leq p < \infty$*;*

(ii) $\alpha_p \leq C^{1/p-1/2}$ *for* $0 < p < 2$*;*

(iii) $\beta_p \leq cp$ *for* $2 \leq p < \infty$*;*

(iv) $\beta_p = 1$ *for* $0 < p < 2$.

A similar result holds for $\mathcal{BMO}^c(\mathcal{M})$ but only with $2 \leq p < \infty$ (see Remark 2.9). On the other hand, our proof of Theorem D can be easily modified to extend Junge/Musat's result to all $0 < p < \infty$ and in the form (1.0.6) (see Corollary 2.19).

We now turn to the second objective of this paper: the atomic decomposition of different noncommutative Hardy spaces. Let us recall the 2-atomic decomposition obtained in [2]. An element $a \in L_1(\mathcal{M})$ is said to be a $(1,2)_c$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and $e \in \mathcal{P}(\mathcal{M}_n)$ such that

(i) $\mathcal{E}_n(a) = 0$; (ii)$ae = a$; (iii) $\|a\|_2 \leq (\tau(e))^{-1/2}$.

The atomic Hardy space $\mathsf{h}_{1,\mathrm{at}}^c(\mathcal{M})$ is defined as the space of all $x \in L_1(\mathcal{M})$, such that the following $\|\cdot\|_{\mathsf{h}_{1,\mathrm{at}}^c}$ norm is finite,

$$\|x\|_{\mathsf{h}_{1,\mathrm{at}}^c} = \|\mathcal{E}_1 x\|_1 + \inf \sum_j |\lambda_j|.$$

Here the infimum is taken for possible decompositions $x - \mathcal{E}_1 x = \sum_j \lambda_j a_j$ with $\lambda_j \in \mathbb{C}$, a_j being $(1,2)_c$-atom. It is proved in [2] that $x \in \mathsf{h}_1^c(\mathcal{M})$ if and only if $x \in \mathsf{h}_{1,\mathrm{at}}^c(\mathcal{M})$ and

$$\|x\|_{\mathsf{h}_1^c} \simeq \|x\|_{\mathsf{h}_{1,\mathrm{at}}^c}.$$

Together with the equivalence $\mathcal{H}_1^c(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M}) + \mathsf{h}_1^d(\mathcal{M})$, the authors of [2] also obtained a 2-atomic decomposition for $\mathcal{H}_1^c(\mathcal{M})$.

Let us briefly recall the argument used in [2]. The dual space of $\mathrm{h}_{1,\mathrm{at}}^c(\mathcal{M})$ can be described as

$$\Lambda^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\Lambda^c} < \infty\}$$

with

$$\|x\|_{\Lambda^c} = \max\{\|\mathcal{E}_1 x\|_\infty, \quad \sup_{n\geq 1} \sup_{e\in\mathcal{P}_n} (\frac{1}{\tau(e)}\tau(e|x - x_n|^2))^{\frac{1}{2}}\}.$$

Actually, the supremum in the definition above can be taken for all $b \in L_1(\mathcal{M}_n)$ since the extreme points of the unit ball of $L_1(\mathcal{M}_n)$ are all multiples of projections. Therefore,

$$\begin{aligned} \|x\|_{\Lambda^c} &= \max\{\|\mathcal{E}_1 x\|_\infty, \quad \sup_{n\geq 1} \sup_{b\in\mathcal{M}_n} (\frac{1}{\|b\|_1}\tau(b|x - x_n|^2))^{\frac{1}{2}}\} \\ &= \max\{\|\mathcal{E}_1 x\|_\infty, \quad \sup_{n\geq 1} \|\mathcal{E}_n|x - x_n|^2\|_\infty^{\frac{1}{2}}\} \\ &= \|x\|_{\mathsf{bmo}^c}. \end{aligned} \tag{1.0.8}$$

Then the duality $\mathsf{h}_1^c(\mathcal{M}) = \mathsf{bmo}^c(\mathcal{M})$ yields $\mathsf{h}_{1,\mathrm{at}}^c(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M})$.

It is well known in the classical theory that 2-atoms in the previous atomic decomposition can be replaced by q-atoms for any $1 < q \leq \infty$. Let us recall these atoms in the commutative case. A function $a \in L_1(\Omega)$ is said to be a q-atom if there exist $n \geq 1$ and $E \in \mathcal{F}_n$ such that

(i) $\mathbb{E}_n a = 0$; (ii) $\{a \neq 0\} \subset E$; (iii) $\|a\|_q \leq \mathbb{P}(E)^{-1+\frac{1}{q}}$.

We refer to [57] for more information.

The main difficulty to obtain q-atomic decompositions in the noncommutative case is that the key equivalence (1.0.8) no longer holds if one replaces the power indices 2 by $q' \neq 2$, $1 \leq q' < \infty$. We overcome this obstacle by Theorem D.

Theorem E. *For all* $1 < q \leq \infty$,

$$\mathsf{h}_1^c(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at_{q,pr}}}^c(\mathcal{M})$$

with equivalent norms. Here $\mathrm{h}_{1,\mathrm{at_{q,pr}}}^{\mathrm{c}}(\mathcal{M})$ *is the* q*-atomic Hardy spaces with its atom* a *defined as: there exist* $n \geq 1$ *and a projection* $e \in \mathcal{P}(\mathcal{M}_n)$ *such that*

(i) $\mathcal{E}_n(a) = 0$;

(ii) $ae = a$;

(iii) $\|a\|_{\mathsf{h}_q^c} \leq (\tau(e))^{-\frac{1}{q'}}$.

Note that $\mathsf{h}_{1,\mathrm{at_{2,pr}}}^c(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at}}^c(\mathcal{M})$, so we recover the 2-atomic decomposition of [2]. Moreover, applying the conditional version of Junge/Musat's theorem in the form (1.0.6), we get a q-atomic decomposition for $\mathsf{h}_1(\mathcal{M})$ in which the atoms are defined in a similar way with $\| \, \|_{\mathsf{h}_q^c}$ in (iii) above replaced by $\| \, \|_q$ and the support condition (ii) weakened to $ae = a$ or $ea = a$ (see Theorem 3.19). This is exactly the noncommutative analogue of the classical atomic decomposition.

This chapter is organized as follows. Section 1.1 is on preliminaries and notation. All the results on John-Nirenberg inequality will be presented in section 1.2. Section 1.3 is devoted to the atomic decomposition of Hardy spaces. In section 1.4, we answer Junge/Musat's question in [23] which implies that the John-Nirenberg inequality in the classical sense does not hold any more in the noncommutative setting.

In this chapter, the letter c always denotes an absolute positive constant, while C an absolute constant bigger than 1. They may vary from lines to lines.

1.1 Preliminaries and notations

Throughout this chapter, we will work on a von Neumann algebra $\mathcal{M}$ with a normal faithful normalized trace τ. For all $0 < p \leq \infty$, let $L_p(\mathcal{M}, \tau)$ or simply $L_p(\mathcal{M})$ be the associated noncommutative L_p spaces. For $x \in L_p(\mathcal{M})$ we denote the right and left supports of x by $r(x)$ and $l(x)$ respectively. $r(x)$ (resp. $l(x)$) is also the least projection e such that $xe = x$ (resp. $ex = x$). If x is selfadjoint, $r(x) = l(x)$, denoted by $s(x)$. We mainly refer the reader to [49] for more information on noncommutative L_p spaces.

Let us recall some basic notions on noncommutative martingales. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of the $\mathcal{M}_n$'s is w^*-dense in $\mathcal{M}$. Let $\mathcal{E}_n$ be the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$. A sequence $x = (x_n)$ in $L_1(\mathcal{M})$ is called a noncommutative martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$ if $\mathcal{E}_n(x_{n+1}) = x_n$ for every $n \geq 1$. If in addition, all the x_n's are in $L_p(\mathcal{M})$ for some $1 \leq p \leq \infty$, x is called an L_p-martingale. In this case we set

$$\|x\|_p = \sup_{n\geq 1} \|x_n\|_p.$$

If $\|x\|_p < \infty$, x is called a bounded L_p-martingale.

Let $x = (x_n)$ be a noncommutative martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$. Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the convention that $x_0 = 0$ and $\mathcal{E}_0 = \mathcal{E}_1$. The sequence

$dx = (dx_n)_n$ is called the martingale difference sequence of x. In the sequel, for any operator $x \in L_1(\mathcal{M})$ we denote $x_n = \mathcal{E}_n(x)$ for $n \geq 1$.

The sequence $(\mathcal{M}_n)_{n\geq 1}$ will be fixed throughout the chapter. All martingales will be with respect to $(\mathcal{M}_n)_{n\geq 1}$. Let $1 \leq p < \infty$. Define $\mathcal{H}_p^c$ (resp. $\mathcal{H}_p^r$) as the completion of all finite L_p-martingales under the norm $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}_p^r} = \|S_r(x)\|_p$), where $S_c(x)$ and $S_r(x)$ are defined as

$$S_c(x) = \big(\sum_{k\geq 1} |dx_k|^2\big)^{1/2}, \quad S_r(x) = S_c(x^*).$$

The noncommutative martingale Hardy spaces $\mathcal{H}_p(\mathcal{M})$ are defined as follows: if $1 \leq p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \inf_{x=y+z}\{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\}.$$

When $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \max\{\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}\}.$$

The space $\mathcal{BMO}^c$ is defined as

$$\mathcal{BMO}^c(\mathcal{M}) = \{x \in L_1(\mathcal{M}) : \|x\|_{\mathcal{BMO}^c} < \infty\}$$

where

$$\|x\|_{\mathcal{BMO}^c} = \sup_{n\geq 1} \|\mathcal{E}_n|x - x_{n-1}|^2\|_\infty^{1/2},$$

and

$$\mathcal{BMO}^r(\mathcal{M}) = \{x : x^* \in \mathcal{BMO}^c(\mathcal{M})\}.$$

Define

$$\mathcal{BMO}(\mathcal{M}) = \mathcal{BMO}^c(\mathcal{M}) \cap \mathcal{BMO}^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{BMO}} = \max\{\|x\|_{\mathcal{BMO}^c}, \|x\|_{\mathcal{BMO}^r}\}.$$

Pisier and Xu [48] proved the two fundamental results: $\mathcal{H}_p(\mathcal{M}) = L_p(\mathcal{M})$ and Fefferman duality between $\mathcal{H}_1(\mathcal{M})$ and $\mathcal{BMO}(\mathcal{M})$. Their work triggered a rapid development of the noncommutative martingale theory.

We will also work on the conditional version of Hardy and BMO spaces developed in [25]. Let $x = (x_n)_{n\geq 1}$ be a finite martingale in $L_2(\mathcal{M})$. We set

$$s_c(x) = \big(\sum_{k\geq 1} \mathcal{E}_{k-1}|dx_k|^2\big)^{1/2} \quad \text{and} \quad s_r(x) = s_c(x^*).$$

Let $0 < p < \infty$. Define $\mathsf{h}_p^c(\mathcal{M})$ (resp. $\mathsf{h}_p^r(\mathcal{M})$) as the completion of all finite L_∞-martingales under the (quasi-)norm $\|x\|_{\mathsf{h}_p^c} = \|s_c(x)\|_p$ (resp. $\|x\|_{\mathsf{h}_p^r} = \|s_r(x)\|_p$). Define $\mathsf{h}_p^d(\mathcal{M})$ as the subspace of $\ell_p(L_p(\mathcal{M}))$ consisting of all martingale difference sequences, where $\ell_p(L_p(\mathcal{M}))$ is the space of all sequences $a = (a_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ such that

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \big(\sum_{n\geq 1} \|a_n\|_p^p\big)^{1/p} < \infty$$

with the usual modification for $p = \infty$. The noncommutative conditional martingale Hardy spaces are defined as follows: if $0 < p < 2$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M}) + \mathsf{h}_p^d(\mathcal{M})$$

equipped with the (quasi-)norm

$$\|x\|_{\mathsf{h}_p} = \inf_{x=y+z+w}\{\|y\|_{\mathsf{h}_p^c} + \|z\|_{\mathsf{h}_p^r} + \|w\|_{\mathsf{h}_p^d}\}.$$

When $2 \leq p < \infty$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^c(\mathcal{M}) \cap \mathsf{h}_p^r(\mathcal{M}) \cap \mathsf{h}_p^d(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathsf{h}_p} = \max\{\|x\|_{\mathsf{h}_p^c}, \|x\|_{\mathsf{h}_p^r}, \|x\|_{\mathsf{h}_p^d}\}.$$

The space bmo^c is defined as

$$\mathsf{bmo}^c(\mathcal{M}) = \{x \in L_1(\mathcal{M}) : \|x\|_{\mathsf{bmo}^c} < \infty\}$$

where

$$\|x\|_{\mathsf{bmo}^c} = \max\left\{\|\mathcal{E}_1(x)\|_\infty, \quad \sup_{n\geq 1} \|\mathcal{E}_n|x - x_n|^2\|_\infty^{1/2}\right\}.$$

Let

$$\mathsf{bmo}^r(\mathcal{M}) = \{x : x^* \in \mathsf{bmo}^c(\mathcal{M})\}.$$

Let $\mathsf{bmo}^d(\mathcal{M})$ be the subspace of $\ell_\infty(L_\infty(\mathcal{M}))$ consisting of all martingale difference sequences. Note that $\mathsf{bmo}^d(\mathcal{M}) = \mathsf{h}_\infty^d(\mathcal{M})$. Define

$$\mathsf{bmo}(\mathcal{M}) = \mathsf{bmo}^c(\mathcal{M}) \cap \mathsf{bmo}^r(\mathcal{M}) \cap \mathsf{bmo}^d(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathsf{bmo}} = \max\{\|x\|_{\mathsf{bmo}^c}, \|x\|_{\mathsf{bmo}^r}, \|x\|_{\mathsf{bmo}^d}\}.$$

We refer to [25], [28], [51], [53], [18], [45] for more information on these spaces.

1.2 John-Nirenberg inequality

1.2.1 A crude version

Definition 2.1. For $0 < p < \infty$, we define

(i)

$$\mathsf{bmo}_p^c(\mathcal{M}) = \big\{x \in L_1(\mathcal{M}) : \|x\|_{\mathsf{bmo}_p^c} < \infty\big\}$$

with

$$\|x\|_{\mathsf{bmo}_p^c} = \max\big\{\|\mathcal{E}_1(x)\|_\infty, \quad \sup_n \sup_{a\in\mathcal{M}_n, \|a\|_p\leq 1,} \|(x - x_n)a\|_{\mathsf{h}_p^c}\big\};$$

(ii)
$$\mathsf{bmo}_p^r(\mathcal{M}) = \{x : x^* \in \mathsf{bmo}_p^c(\mathcal{M})\};$$

(iii)
$$\mathsf{bmo}_p(\mathcal{M}) = \mathsf{bmo}_p^c(\mathcal{M}) \cap \mathsf{bmo}_p^r(\mathcal{M}) \cap \mathsf{bmo}^d(\mathcal{M})$$

equipped with the (quasi-)norm
$$\|x\|_{\mathsf{bmo}_p} = \max\{\|x\|_{\mathsf{bmo}_p^c}, \|x\|_{\mathsf{bmo}_p^r}, \|x\|_{\mathsf{bmo}^d}\}.$$

Remark 2.2. When $p = 2$, these are exactly the spaces $\mathsf{bmo}^c(\mathcal{M})$, $\mathsf{bmo}^r(\mathcal{M})$ and $\mathsf{bmo}(\mathcal{M})$.

Below is our first version of the column (resp. row) John-Nirenberg inequality.

Theorem 2.3. *For all $0 < p < \infty$, there exist two constants α_p and β_p such that*
$$\alpha_p^{-1}\|x\|_{\mathsf{bmo}^c} \leq \|x\|_{\mathsf{bmo}_p^c} \leq \beta_p\|x\|_{\mathsf{bmo}^c},$$
with α_p and β_p satisfying

(i) $\alpha_p = 1$ *for* $2 \leq p < \infty$*;*

(ii) $\alpha_p \leq C^{1/p-1/2}$ *for* $0 < p < 2$*;*

(iii) $\beta_p \leq cp$ *for* $2 \leq p < \infty$*;*

(iv) $\beta_p = 1$ *for* $0 < p < 2$.

The similar inequalities hold for $\|\cdot\|_{\mathsf{bmo}_p^r}$ and $\|\cdot\|_{\mathsf{bmo}^r}$.

Proof. We only need to prove the column case, since the row case can be done by replacing x with x^*. First consider the case $2 < p < \infty$. We will show the following inequalities:
$$\|x\|_{\mathsf{bmo}_2^c} \leq \|x\|_{\mathsf{bmo}_p^c} \leq cp\|x\|_{\mathsf{bmo}_2^c}.$$
The left inequality is obtained directly by Hölder's inequality. In fact, taking $a \in \mathcal{M}_n$ with $\|a\|_2 \leq 1$, there exists a factorization $a = a_0a_1$ such that $\|a_0\|_p = \|a\|_2^{2/p} \leq 1$ and $\|a_1\|_{2p/(p-2)} = \|a\|_2^{(p-2)/p} \leq 1$, so
$$\begin{aligned}\|(x-x_n)a\|_{\mathsf{h}_2^c}^2 &= \tau(a_1^*a_0^*s_c^2(x-x_n)a_0a_1)\\ &\leq \|a_1^*\|_{\frac{2p}{p-2}}\|a_0^*s_c^2(x-x_n)a_0\|_{\frac{p}{2}}\|a_1\|_{\frac{2p}{p-2}}\\ &\leq \|(x-x_n)a_0\|_{\mathsf{h}_p^c}^2.\end{aligned}$$
We invoke complex interpolation to prove the right inequality. Fix n, let $b \in L_p(\mathcal{M}_n)$ with $\|b\|_p \leq 1$ and $S = \{z \in \mathbb{C} : 0 \leq Rez \leq 1\}$. Then by interpolation between L_p spaces $L_p = (L_2, L_\infty)_\theta$, there exists an operator-valued function B which is continuous on S and analytic in the interior of S such that $B(\theta) = b$ and
$$\sup_{t\in\mathbb{R}}\|B(it)\|_2 \leq 1, \qquad \sup_{t\in\mathbb{R}}\|B(1+it)\|_\infty \leq 1.$$
Define
$$f(z) = (x - x_n)B(z).$$

Then on the one hand, by the definition of $\mathsf{bmo}_2^c(\mathcal{M})$, we have

$$\|f(it)\|_{\mathsf{h}_2^c} \leq \|x\|_{\mathsf{bmo}_2^c}.$$

On the other hand, by a simple calculation, we have

$$\|f(1+it)\|_{\mathsf{bmo}_2^c} \leq \|x-x_n\|_{\mathsf{bmo}_2^c}\|B(1+it)\|_\infty \leq \|x\|_{\mathsf{bmo}_2^c}.$$

Therefore, by interpolation,

$$\|f(\theta)\|_{(\mathsf{h}_2^c,\mathsf{bmo}^c)_\theta} \leq \|x\|_{\mathsf{bmo}_2^c} = \|x\|_{\mathsf{bmo}^c}.$$

However by [2],

$$(\mathsf{h}_2^c, \mathsf{bmo}^c)_\theta \subset \mathsf{h}_p^c$$

with relevant constant majorized by cp. We then deduce that

$$\|f(\theta)\|_{\mathsf{h}_p^c} \leq cp\|x\|_{\mathsf{bmo}^c}, \tag{1.2.1}$$

hence the desired inequality holds.

For the case $0 < p < 2$. We show the following inequalities:

$$\|x\|_{\mathsf{bmo}_p^c} \leq \|x\|_{\mathsf{bmo}_2^c} \leq C^{1/p-1/2}\|x\|_{\mathsf{bmo}_p^c}.$$

Again, the left inequality is obtained by Hölder's inequality. It remains to prove the right one. We choose $2 < p_1 < \infty$ and $0 < \theta < 1$ such that $1/2 = (1-\theta)/p + \theta/p_1$. Fix n, by the definition of $\mathsf{bmo}_p^c(\mathcal{M})$, we can view $x - x_n$ as a bounded operator from $L_p(\mathcal{M}_n)$ to $\mathsf{h}_p^c(\mathcal{M})$. Then we have the following two inequalities:

$$\|x-x_n\|_{L_p(\mathcal{M}_n)\to\mathsf{h}_p^c} \leq \|x\|_{\mathsf{bmo}_p^c}, \qquad \|x-x_n\|_{L_{p_1}(\mathcal{M}_n)\to\mathsf{h}_{p_1}^c} \leq \|x\|_{\mathsf{bmo}_{p_1}^c}.$$

Then by interpolation, we get

$$\|x-x_n\|_{L_2(\mathcal{M}_n)\to(\mathsf{h}_p^c,\mathsf{h}_{p_1}^c)_\theta} \leq \|x\|_{\mathsf{bmo}_p^c}^{1-\theta}\|x\|_{\mathsf{bmo}_{p_1}^c}^{\theta}.$$

Now by the trivial contractive inclusion $(\mathsf{h}_p^c, \mathsf{h}_{p_1}^c)_\theta \subset \mathsf{h}_2^c$, and the right inequality in the case $2 < p_1 < \infty$, we get

$$\|x-x_n\|_{L_2(\mathcal{M}_n)\to\mathsf{h}_2^c} \leq cp_1\|x\|_{\mathsf{bmo}_p^c}^{1-\theta}\|x\|_{\mathsf{bmo}_2^c}^{\theta}.$$

Therefore,

$$\|x\|_{\mathsf{bmo}_2^c} \leq (cp_1)^\theta\|x\|_{\mathsf{bmo}_p^c}^{1-\theta}\|x\|_{\mathsf{bmo}_2^c}^{\theta},$$

hence

$$\|x\|_{\mathsf{bmo}_2^c} \leq (cp_1)^{\frac{\theta}{1-\theta}}\|x\|_{\mathsf{bmo}_p^c}.$$

Noting that $\theta/(1-\theta) = (1/p-1/2)/(1/2-1/p_1)$, we get the desired estimate by taking $C = (cp_1)^{1/(1/2-1/p_1)}$. $\square$

Remark 2.4. The constant in (1.2.1) is optimal. This can be seen as follows. By Lemma 4.3 in [2], $\mathsf{h}_{p'}^c(\mathcal{M})$ embeds into $(\mathsf{h}_2^c(\mathcal{M}), \mathsf{h}_1^c(\mathcal{M}))_\theta$ with constant independent of p'. So $((\mathsf{h}_2^c(\mathcal{M}))^*, (\mathsf{h}_1^c(\mathcal{M}))^*)_\theta$ embeds into $(\mathsf{h}_{p'}^c(\mathcal{M}))^*$ with constant independent of p by duality. Finally, by the optimal embedding $(\mathsf{h}_{p'}^c(\mathcal{M}))^* \subset \mathsf{h}_p^c(\mathcal{M})$ with constant cp in [25] and $\mathsf{bmo}^c(\mathcal{M}) \subset (\mathsf{h}_1^c(\mathcal{M}))^*$ in [45], $(\mathsf{h}_2^c(\mathcal{M}), \mathsf{bmo}^c(\mathcal{M}))_\theta$ embeds into $\mathsf{h}_p^c(\mathcal{M})$ with optimal constant cp.

It is natural to ask whether there is a result similar to Theorem 2.3 for $\mathcal{BMO}^c$ by replacing h_p^c and $x - x_n$ in the definition of bmo_p^c by $\mathcal{H}_p^c$ and $x - x_{n-1}$ respectively. Using the identity

$$\mathcal{BMO}^c(\mathcal{M}) \simeq \mathsf{bmo}^c(\mathcal{M}) \cap \mathsf{bmo}^d(\mathcal{M})$$

proved in [45], we are reduced to deal with the diagonal space $\mathsf{bmo}^d(\mathcal{M})$. Surprisingly, the result is true only for $2 \leq p < \infty$ (see Remark 2.9).

Definition 2.5. For $1 \leq p < \infty$, we define

(i)
$$\mathcal{BMO}_p^c(\mathcal{M}) = \Big\{ x \in L_1(\mathcal{M}) : \|x\|_{\mathcal{BMO}_p^c} < \infty \Big\}$$
with
$$\|x\|_{\mathcal{BMO}_p^c} = \sup_n \sup_{a \in \mathcal{M}_n, \|a\|_p \leq 1} \|(x - x_{n-1})a\|_{\mathcal{H}_p^c};$$

(ii)
$$\mathcal{BMO}_p^r(\mathcal{M}) = \{x : x^* \in \mathcal{BMO}_p^c(\mathcal{M})\};$$

(iii)
$$\mathcal{BMO}_p(\mathcal{M}) = \mathcal{BMO}_p^c(\mathcal{M}) \cap \mathcal{BMO}_p^r(\mathcal{M})$$
equipped with the norm
$$\|x\|_{\mathcal{BMO}_p} = \max\{\|x\|_{\mathcal{BMO}_p^c}, \|x\|_{\mathcal{BMO}_p^r}\}.$$

Remark 2.6. For $p = 2$, we recover the spaces $\mathcal{BMO}^c(\mathcal{M})$, $\mathcal{BMO}^r(\mathcal{M})$ and $\mathcal{BMO}(\mathcal{M})$.

The following lemma will alow us to handle with the diagonal space $\mathsf{bmo}^d(\mathcal{M})$.

Lemma 2.7. *For $2 \leq p < \infty$, we have*

$$cp^{-1}\|b\|_\infty \leq \sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|ba\|_{\mathcal{H}_p^c} \leq cp^{\frac{1}{2}}\|b\|_\infty.$$

Proof. Note that $\|\cdot\|_{\mathcal{H}_p^c} \leq cp^{1/2}\|\cdot\|_p$ (see [51], Remark 5.4 as a reference for the constant we use here), we have

$$\sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|ba\|_{\mathcal{H}_p^c} \leq cp^{\frac{1}{2}} \sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|ba\|_p = cp^{\frac{1}{2}}\|b\|_\infty.$$

For the first inequality, without loss of generality assume $\|b\|_\infty = 1$. Note that for selfadjoint $x \in \mathcal{M}$, $\|x\|_p \leq cp\|x\|_{\mathcal{H}_p^c}$ (see [51], Remark 5.4). Then

$$\begin{aligned}
\|b^*\|_\infty &= \sup_{y \in \mathcal{M}, \|y\|_{2p} \leq 1} \|yb^*\|_{2p} \\
&= \sup_{y \in \mathcal{M}, \|y\|_{2p} \leq 1} \|b|y|^2 b^*\|_p^{\frac{1}{2}} \\
&\leq cp^{\frac{1}{2}} \sup_{y \in \mathcal{M}, \|y\|_{2p} \leq 1} \|b|y|^2 b^*\|_{\mathcal{H}_p^c}^{\frac{1}{2}} \\
&\leq cp^{\frac{1}{2}} \sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|ba\|_{\mathcal{H}_p^c}^{\frac{1}{2}}.
\end{aligned}$$

And then $cp^{-1}\|b\|_\infty \leq \sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|ba\|_{\mathcal{H}_p^c}$. $\square$

Theorem 2.8. *For all $2 \leq p < \infty$, we have*

$$\mathcal{BMO}_p^c(\mathcal{M}) = \mathcal{BMO}^c(\mathcal{M})$$

with equivalent norms. More precisely,

$$cp^{-1}\|x\|_{\mathcal{BMO}^c} \leq \|x\|_{\mathcal{BMO}_p^c} \leq cp\|x\|_{\mathcal{BMO}^c}.$$

Similarly, $\mathcal{BMO}_p^r(\mathcal{M}) = \mathcal{BMO}^r(\mathcal{M})$ with equivalent norms.

Using the previous lemma and the identity $\mathcal{BMO}^c(\mathcal{M}) \simeq \mathsf{bmo}^c(\mathcal{M}) \cap \mathsf{bmo}^d(\mathcal{M})$, we can easily deduce Theorem 2.8 from Theorem 2.3. We will however present a direct proof.
Proof. We only prove the inequalities for the column case, the row case can be dealt with similarly. By the previous lemma and Hölder's inequality, we have

$$\begin{aligned}
\|\mathcal{E}_n \sum_{k=n}^{\infty} |dx_k|^2\|_\infty &\leq \sup_{b\in\mathcal{M}_n^+,\|b\|_1\leq 1} \tau\left(\sum_{k=n+1}^{\infty} |dx_k|^2 b\right) + \|x_n - x_{n-1}\|_\infty^2 \\
&\leq \sup_{b\in\mathcal{M}_n^+,\|b\|_1\leq 1} \tau\left(\sum_{k=n+1}^{\infty} |(dx_k)b^{\frac{1}{p}}|^2 b^{\frac{p-2}{p}}\right) \\
&\qquad + cp^2 \sup_{a\in\mathcal{M}_n,\|a\|_p\leq 1} \|(x_n - x_{n-1})a\|_{\mathcal{H}_p^c}^2 \\
&\leq \sup_{b\in\mathcal{M}_n^+,\|b\|_1\leq 1} \left\|\sum_{k=n+1}^{\infty} |(dx_k)b^{\frac{1}{p}}|^2\right\|_{\frac{p}{2}} \left\|b^{\frac{p-2}{p}}\right\|_{(\frac{p}{2})'} \\
&\qquad + cp^2 \sup_{a\in\mathcal{M}_n,\|a\|_p\leq 1} \|(x_n - x_{n-1})a\|_{\mathcal{H}_p^c}^2 \\
&\leq \sup_{b\in\mathcal{M}_n^+,\|b\|_1\leq 1} \left\|(x - x_n)b^{\frac{1}{p}}\right\|_{\mathcal{H}_p^c}^2 \\
&\qquad + cp^2 \sup_{a\in\mathcal{M}_n,\|a\|_p\leq 1} \|(x_n - x_{n-1})a\|_{\mathcal{H}_p^c}^2.
\end{aligned}$$

Then by $\|\mathcal{E}_n x\|_{\mathcal{H}_p^c} \leq \|x\|_{\mathcal{H}_p^c}$,

$$\|x\|_{\mathcal{BMO}_2^c} \leq cp \sup_{a\in\mathcal{M}_n,\|a\|_p\leq 1} \|(x - x_{n-1})a\|_{\mathcal{H}_p^c} = cp\|x\|_{\mathcal{BMO}_p^c}.$$

Conversely, by the previous lemma,

$$\begin{aligned}
\|x\|_{\mathcal{BMO}_p^c} &\leq \sup_n \sup_{a\in\mathcal{M}_n,\|a\|_p\leq 1} \|(x - x_n)a\|_{\mathcal{H}_p^c} \\
&\qquad + \sup_n \sup_{a\in\mathcal{M}_n,\|a\|_p\leq 1} \|(x_n - x_{n-1})a\|_{\mathcal{H}_p^c} \\
&\leq \sup_n \sup_{a\in\mathcal{M}_n,\|a\|_p\leq 1} \|(x - x_n)a\|_{\mathcal{H}_p^c} + cp^{\frac{1}{2}} \sup_n \|x_n - x_{n-1}\|_\infty \\
&\leq \sup_n \sup_{a\in\mathcal{M}_n,\|a\|_p\leq 1} \left\|(dx_k a)_{k=n+1}^\infty\right\|_{L_p(\ell_2^c)} + cp^{\frac{1}{2}}\|x\|_{\mathcal{BMO}_2^c}. \qquad (1.2.2)
\end{aligned}$$

Note that, by the Hahn-Banach theorem and the duality between $\mathcal{H}_1^c(\mathcal{M})$ and $\mathcal{BMO}^c(\mathcal{M})$, there exists a sequence $(b_n)_{n=1}^\infty \in L_\infty(\mathcal{M};\ell_2^c)$ such that

$$\|(b_n)_{n=1}^\infty\|_{L_\infty(\ell_2^c)} = \|x\|_{\mathcal{BMO}^c}, \quad dx_k = \mathcal{E}_k b_k - \mathcal{E}_{k-1} b_k.$$

Thus by the noncommutative Stein inequality (see [51] for the constant used below) and Hölder's inequality,

$$\begin{aligned}
&\sup_{a\in\mathcal{M}_n,\|a\|_p\le 1}\left\|(dx_k a)_{k=n+1}^{\infty}\right\|_{L_p(\ell_2^c)}\\
&\quad\le \sup_{a\in\mathcal{M}_n,\|a\|_p\le 1}\left\|(\mathcal{E}_k(b_k a))_{k=n+1}^{\infty}\right\|_{L_p(\ell_2^c)}\\
&\qquad\quad+\sup_{a\in\mathcal{M}_n,\|a\|_p\le 1}\|(\mathcal{E}_k b_{k+1} a)_{k=n}^{\infty}\|_{L_p(\ell_2^c)}\\
&\quad\le cp\sup_{a\in\mathcal{M}_n,\|a\|_p\le 1}\left\|(b_k a)_{k=n+1}^{\infty}\right\|_{L_p(\ell_2^c)}\\
&\quad\le cp\left\|\sum_{k=1}^{\infty}|b_k|^2\right\|_{\infty}^{\frac12}=cp\|x\|_{\mathcal{BMO}_2^c}.
\end{aligned}$$

Combining this with (1.2.2) we finish the proof. □

Remark 2.9. It is a bit surprising that Theorem 2.8 is actually wrong for any $p<2$. Indeed, choose a filtration $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$,...,$\mathcal{M}_{n-1}$ and $y\in\mathcal{M}_{n-1}$ such that $\|y\|_p=1$ and $\|y\|_{\mathcal{H}_p^c}=c_n>>1$. Let $\mathcal{M}_n=L_\infty(\Omega,\mathcal{M}_{n-1})$ with $\Omega=\{0,1\}$ with $\mu\{1\}=\mu\{0\}=1/2$. We certainly can view $\mathcal{M}_k$, $k<n$ as the space of constant functions on Ω, so $\mathcal{M}_k\subset\mathcal{M}_n$. Let $x=1$ on $\{0\}$ and $x=-1$ on $\{1\}$ then $x_{n-1}=0$. Let $a=y$ on $\{0\}$ and $a=-y$ on $\{1\}$. Then $(x-x_{n-1})a=y$ whose $\mathcal{H}_p^c$ norm equals c_n and $\|a\|_p=1$, so $\|x\|_{\mathcal{BMO}_p^c}\ge c_n$. But $\|x\|_{\mathcal{BMO}_2^c}=1$.

In the rest of this subsection, we turn to Junge/Musat's type of John-Nirenberg inequality. In [23], Junge and Musat established the inequality for $2<p<\infty$ in the state case. Later the second author of the present paper gave a simple proof for all $1\le p<\infty$ in the tracial setting (see [33]). The idea of the proof of Theorem 2.3 can be applied to obtain this inequality for all $0<p<\infty$ (see Corollary 2.13). We start again with $\mathsf{bmo}(\mathcal{M})$.

Theorem 2.10. *For all $0<p<\infty$, we have*

$$\alpha_p^{-1}\|x\|_{\mathsf{bmo}}\le \mathsf{b}_p(x)\le\beta_p\|x\|_{\mathsf{bmo}}$$

where

$$\begin{aligned}
\mathsf{b}_p(x)=\max\{&\sup_n\|(dx_n)_n\|_\infty,\ \sup_n\sup_{b\in\mathcal{M}_n,\|b\|_p\le 1}\|(x-x_n)b\|_p,\\
&\sup_n\sup_{b\in\mathcal{M}_n,\|b\|_p\le 1}\|b(x-x_n)\|_p\}.
\end{aligned}$$

The constant α_p and β_p have the same orders as those in Theorem 2.3.

Proof. We first treat the case $2\le p<\infty$. For $p=2$, it is trivial. So we can assume $2<p<\infty$. The inequality

$$\|x\|_{\mathsf{bmo}}\le\mathsf{b}_p(x)$$

follows from Hölder's inequality. We will prove the reverse inequality by interpolation. By a simple calculation, we have the following estimates

$$\|(x-x_n)b\|_{\mathsf{bmo}^c}\le\|x\|_{\mathsf{bmo}^c}\|b\|_\infty,$$

$$\|(x-x_n)b\|_{\mathsf{bmo}^r}\le\|x\|_{\mathsf{bmo}^r}\|b\|_\infty,$$

$$\|(x-x_n)b\|_{\mathsf{bmo}^d} \le \|x\|_{\mathsf{bmo}^d}\|b\|_\infty.$$

Then it follows that

$$\|(x-x_n)b\|_{\mathsf{bmo}} \le \|x\|_{\mathsf{bmo}}\|b\|_\infty.$$

On the other hand, it is clear that

$$\|(x-x_n)b\|_2 = \|(x-x_n)b\|_{\mathsf{h}_2^c} \le \|x\|_{\mathsf{bmo}}\|b\|_2.$$

Then by the interpolation result of [2], we have

$$\begin{aligned}\|(x-x_n)b\|_p &\le cp\|(x-x_n)b\|_{(L_2,\mathsf{bmo})_\theta} \\ &\le cp\|x\|_{\mathsf{bmo}}\|b\|_p.\end{aligned} \tag{1.2.3}$$

In the same way, we obtain

$$\|b(x-x_n)\|_p \le cp\|x\|_{\mathsf{bmo}}\|b\|_p.$$

Thus we prove the assertion.

Now we turn to the case $0 < p < 2$, by Hölder's inequality, we obtain the trivial part

$$\mathsf{b}_p(x) \le \mathsf{b}_2(x) = \|x\|_{\mathsf{bmo}}.$$

Let us prove the inverse one, let $2 < p_1 < \infty$ and θ be such that

$$\frac{1}{2} = \frac{1-\theta}{p} + \frac{\theta}{p_1}.$$

We view $x - x_n$ and $(x-x_n)^*$ as two operators. By interpolation,

$$\begin{aligned}&\|(x-x_n)\|_{L_2(\mathcal{M}_n)\to L_2(\mathcal{M})} \\ &\le \|(x-x_n)\|^{1-\theta}_{L_p(\mathcal{M}_n)\to L_p(\mathcal{M})}\|(x-x_n)\|^{\theta}_{L_{p_1}(\mathcal{M}_n)\to L_{p_1}(\mathcal{M})}\end{aligned}$$

and similarly for $(x-x_n)^*$. By the estimate for $p_1 > 2$, we have

$$\mathsf{b}_2(x) \le (cp_1)^\theta \mathsf{b}_p^{1-\theta}(x)\mathsf{b}_2^\theta(x).$$

Therefore, we obtain

$$\|x\|_{\mathsf{bmo}} \le (cp_1)^{\frac{\theta}{1-\theta}}\mathsf{b}_p(x) = C^{1/p-1/2}\mathsf{b}_p(x),$$

with $C = (cp_1)^{1/(1/2-1/p_1)}$. □

Remark 2.11. The constant in (1.2.3) is optimal. This can be seen as follows. By Lemma 4.3 in [2], $\mathsf{h}^c_{p'}(\mathcal{M})$ embeds into $(\mathsf{h}^c_2(\mathcal{M}),\mathsf{h}^c_1(\mathcal{M}))_\theta$ with constant independent of p'. So $\mathsf{h}_{p'}(\mathcal{M})$ embeds into $(\mathsf{h}_2(\mathcal{M}),\mathsf{h}_1(\mathcal{M}))_\theta$ with constant independent of p'. Now by Theorem 4.1 in [53], $L_{p'}(\mathcal{M})$ embeds into $\mathsf{h}_{p'}(\mathcal{M})$, hence into $(\mathsf{h}_2(\mathcal{M}),\mathsf{h}_1(\mathcal{M}))_\theta$ with optimal constant $c/(p'-1)$. Then by duality, $((\mathsf{h}_2(\mathcal{M}))^*,(\mathsf{h}_1(\mathcal{M}))^*)_\theta$ embeds into $(L_{p'}(\mathcal{M}))^* = L_p(\mathcal{M})$ with best constant cp. At last, by $\mathsf{bmo}(\mathcal{M}) \subset (\mathsf{h}_1(\mathcal{M}))^*$ in [45], $(\mathsf{h}_2(\mathcal{M}),\mathsf{bmo}(\mathcal{M}))_\theta$ embeds into $L_p(\mathcal{M})$ with optimal constant cp.

Remark 2.12. We can directly compare the norms $\|\cdot\|_{\mathsf{bmo}_p}$ and $\mathsf{b}_p(\cdot)$ directly for $1 < p < \infty$ by using Theorem 2.3.

Let us justify this remark. We first deal with the case $2 < p < \infty$. Fix n, for any $b \in \mathcal{M}_n$ with $\|b\|_p \le 1$, by the noncommutative Burkholder inequality [25], we have

$$\|(x - x_n)b\|_{\mathsf{h}_p^c} \le cp\|(x - x_n)b\|_p, \quad \|b(x - x_n)\|_{\mathsf{h}_p^r} \le cp\|b(x - x_n)\|_p,$$

hence

$$\|(x - x_n)b\|_{\mathsf{h}_p^c},\ \|b(x - x_n)\|_{\mathsf{h}_p^r} \le cp\mathsf{b}_p(x)$$

Then by Theorem 2.3,

$$\|x\|_{\mathsf{bmo}_p} \le cp\mathsf{b}_p(x).$$

Another direction can be done by the way in Theorem 2.10,

$$\mathsf{b}_p(x) \le cp\|x\|_{\mathsf{bmo}} \le cp\|x\|_{\mathsf{bmo}_p}.$$

For the case $1 < p < 2$. The trivial part

$$\mathsf{b}_p(x) \le c\|x\|_{\mathsf{bmo}_p}$$

follows from the noncommutative Burkholder inequality in [25]. Now let us prove the inverse one. Take $b \in \mathcal{M}_n$ with $\|b\|_2 \le 1$. By Hölder's inequality, we have

$$\begin{aligned}\|(x - x_n)b\|_2^2 &= \tau(b^{2/p'}(x - x_n)^*(x - x_n)b^{2/p}) \\ &\le \|b^{2/p'}(x - x_n)^*\|_{p'}\|(x - x_n)b^{2/p}\|_p\end{aligned}$$

and

$$\begin{aligned}\|b(x - x_n)\|_2^2 &= \tau((x - x_n)^*b^{2/p'}b^{2/p}(x - x_n)) \\ &\le \|(x - x_n)^*b^{2/p'}\|_{p'}\|b^{2/p}(x - x_n)\|_p.\end{aligned}$$

So by the result in Theorem 2.3 for $2 < p' < \infty$, we have

$$\begin{aligned}&\|b(x - x_n)\|_2^2, \|(x - x_n)b\|_2^2 \\ &\le \max\{\|b^{2/p'}(x - x_n)^*\|_{p'}, \|(x - x_n)^*b^{2/p'}\|_{p'}\} \\ &\quad \cdot \max\{\|(x - x_n)b^{2/p}\|_p, \|b^{2/p}(x - x_n)\|_p\} \\ &\le c\|x\|_{\mathsf{bmo}_{p'}} \cdot \mathsf{b}_p(x) \le cp'\|x\|_{\mathsf{bmo}_2} \cdot \mathsf{b}_p(x)\end{aligned}$$

Then by the definition of $\mathsf{bmo}_2(\mathcal{M})$, we finish the proof by Theorem 2.3

$$\|x\|_{\mathsf{bmo}_p} \le \|x\|_{\mathsf{bmo}_2} \le cp'\mathsf{b}_p(x).$$

The following corollary extends Junge/Musat's theorem to all $0 < p < \infty$. It can be proved similarly as Theorem 2.3. However, using the identity $\mathcal{BMO}(\mathcal{M}) \simeq \mathsf{bmo}(\mathcal{M})$ proved in [45], we give a simpler proof.

Corollary 2.13. *For $0 < p < \infty$, we have*

$$\alpha_p^{-1}\|x\|_{\mathcal{BMO}} \le \mathcal{B}_p(x) \le \beta_p\|x\|_{\mathcal{BMO}},$$

where

$$\begin{aligned}\mathcal{B}_p(x) = \max\{&\sup_n \sup_{b\in\mathcal{M}_n, \|b\|_p\le 1} \|(x - x_{n-1})b\|_p, \\ &\sup_n \sup_{b\in\mathcal{M}_n, \|b\|_p\le 1} \|b(x - x_{n-1})\|_p\}.\end{aligned}$$

The constant α_p and β_p have the same orders as those in Theorem 2.3.

Proof. For $2 \leq p < \infty$, it is very easy to get

$$\mathcal{B}_p(x) \leq \mathrm{b}_p(x) \leq cp\|x\|_{\mathsf{bmo}} \leq cp\|x\|_{\mathcal{BMO}}$$

from the triangular inequality

$$\|(x - x_{n-1})b\|_p \leq \|(x - x_n)b\|_p + \|(x_n - x_{n-1})b\|_p,$$

with $b \in \mathcal{M}_n$ and $\|b\|_p \leq 1$. And the rest of the proof is the same to Theorem 2.10. $\square$

Remark 2.14. The following example shows that Junge/Musat's John-Nirenberg inequality does not hold for bmo^c or $\mathcal{BMO}^c$. The example is the same as the one given in Remark 3.20 of [23]. Let n be a positive integer and consider the von Neumann algebra

$$\mathcal{M} = L_\infty(\mathbb{T})\bar{\otimes}M_n,$$

where M_n is the algebra of $n \times n$ matrices with normalized trace. For $k \geq 1$ let $\mathcal{F}_k$ be the σ-algebra generated by dyadic intervals in $\mathbb{T}$ of length 2^{-k}. Denote by $\mathcal{M}_k$ the subalgebra $L_\infty(\mathbb{T}, \mathcal{F}_k)\bar{\otimes}M_n$ of $\mathcal{M}$ and let $\mathcal{E}_k = \mathbb{E}_k \otimes id_{M_n}$ be the conditional expectation onto $\mathcal{M}_k$. Let r_k be the k-th Rademacher function on $\mathbb{T}$ and consider

$$x = \sum_{k=1}^{n} r_k \otimes e_{1k}.$$

Then x is a martingale relative to the filtration $(\mathcal{M}_k)_{k\geq 1}$ and the martingale differences are given by $dx_k = r_k \otimes e_{1k}$. A simple calculation shows that

$$\sup_m \|x - x_m\|_p = (n-1)^{\frac{1}{2}} n^{-\frac{1}{p}},$$

while

$$\|x\|_{\mathsf{bmo}^c} = \sup_m \Big\| \sum_{k=m+1}^{n} \mathcal{E}_m |d_k x|^2 \Big\|_\infty^{\frac{1}{2}} = 1.$$

Let $p > 2$. Then for any $c > 0$, there exists $n \geq 1$ such that $(n-1)^{1/2} n^{-1/p} > c$. Hence

$$\sup_m \sup_{b \in \mathcal{M}_m, \|b\|_p \leq 1} \|(x - x_m)b\|_p \geq \sup_m \|x - x_m\|_p >> \|x\|_{\mathsf{bmo}^c}.$$

1.2.2 A fine version

Now we can formulate the fine version of the column (resp. row) John-Nirenberg inequality.

Definition 2.15. For $0 < p < \infty$, we define

$$\mathsf{bmo}^c_{p,\mathrm{pr}}(\mathcal{M}) = \big\{x \in L_1(\mathcal{M}) : \|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} < \infty\big\}$$

with

$$\|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} = \max\Big\{\|\mathcal{E}_1(x)\|_\infty, \quad \sup_n \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \|(x - x_n)\frac{e}{(\tau(e))^{1/p}}\|_{\mathsf{h}^c_p}\Big\}.$$

Similarly,

$$\mathsf{bmo}^r_{p,\mathrm{pr}}(\mathcal{M}) = \{x : x^* \in \mathsf{bmo}^c_{p,\mathrm{pr}}(\mathcal{M})\} \;\text{ with }\; \|x\|_{\mathsf{bmo}^r_{p,\mathrm{pr}}} = \|x^*\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}}.$$

Finally,

$$\mathsf{bmo}_{p,\mathrm{pr}}(\mathcal{M}) = \mathsf{bmo}^c_{p,\mathrm{pr}}(\mathcal{M}) \cap \mathsf{bmo}^r_{p,\mathrm{pr}}(\mathcal{M}) \cap \mathsf{bmo}^d(\mathcal{M})$$

equipped with

$$\|x\|_{\mathsf{bmo}_{p,\mathrm{pr}}} = \max\{\|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}}, \|x\|_{\mathsf{bmo}^r_{p,\mathrm{pr}}}, \|x\|_{\mathsf{bmo}^d}\}.$$

The fine version of the column (resp. row) John-Nirenberg inequality is stated as follows.

Theorem 2.16. *For all* $0 < p < \infty$*, we have*

$$\alpha_p^{-1}\|x\|_{\mathsf{bmo}^c} \leq \|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} \leq \beta_p\|x\|_{\mathsf{bmo}^c}.$$

The constants α_p *and* β_p *have the same properties as those in Theorem 2.3. The same inequalities hold for* $\|\cdot\|_{\mathsf{bmo}^r}$ *and* $\|\cdot\|_{\mathsf{bmo}^r_{p,\mathrm{pr}}}$.

Proof. We first consider the case $0 < p \leq 1$. By Theorem 2.3, the trivial part

$$\|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} \leq \|x\|_{\mathsf{bmo}^c_p} \leq \|x\|_{\mathrm{bmo}^c}$$

follows from the fact that $e/(\tau(e))^{1/p} \in \mathcal{M}_n$ and its L_p-norm equals 1. Now we turn to the proof of the inverse inequality. Since any $a \in \mathcal{M}_n$ with $\|a\|_p \leq 1$ can be approximated by sums $\sum_k \lambda_k e_k/(\tau(e_k))^{1/p}$ with e_k's in $\mathcal{M}_n$ and $\sum_k |\lambda_k|^p \leq 1$. Thus we can assume that a itself is such a sum. Then

$$\begin{aligned}\|(x-x_n)a\|^p_{\mathsf{h}^c_p} &= \|\sum_k \lambda_k(x-x_n)\frac{e_k}{(\tau(e_k))^{1/p}}\|^p_{\mathsf{h}^c_p}\\ &\leq \sum_k |\lambda_k|^p\|(x-x_n)\frac{e_k}{(\tau(e_k))^{1/p}}\|^p_{\mathsf{h}^c_p}\\ &\leq \sum_k |\lambda_k|^p\|x\|^p_{\mathsf{bmo}^c_{p,\mathrm{pr}}} \leq \|x\|^p_{\mathsf{bmo}^c_{p,\mathrm{pr}}}.\end{aligned}$$

Therefore by Theorem 2.3,

$$\|x\|_{\mathsf{bmo}^c} \leq C^{1/p-1/2}\|x\|_{\mathsf{bmo}^c_p} \leq C^{1/p-1/2}\|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}}.$$

Now let $1 < p < \infty$. Again, because of the fact that $e/(\tau(e))^{1/p} \in \mathcal{M}_n$ and its L_p-norm equals 1, by Theorem 2.3,

$$\|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}} \leq \|x\|_{\mathsf{bmo}^c_p} \leq c_1 p\|x\|_{\mathsf{bmo}^c}. \tag{1.2.4}$$

We exploit the result for $p = 1$ to prove the inverse inequality. By Hölder's inequality, we have

$$\|x\|_{\mathsf{bmo}^c_{1,\mathrm{pr}}} \leq \|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}}.$$

We end the proof by Theorem 2.3 and the result for $p = 1$,

$$\|x\|_{\mathsf{bmo}^c} \leq C^{1/p-1/2}\|x\|_{\mathsf{bmo}^c_1} \leq C^{1/p-1/2}\|x\|_{\mathsf{bmo}^c_{1,\mathrm{pr}}} \leq C^{1/p-1/2}\|x\|_{\mathsf{bmo}^c_{p,\mathrm{pr}}}.$$

□

Now we give the distributional form of the John-Nirenberg inequality for $\mathsf{bmo}^c(\mathcal{M})$ and $\mathsf{bmo}^r(\mathcal{M})$.

Theorem 2.17. *Let* $x \in \mathsf{bmo}^c(\mathcal{M})$*. Then for all natural numbers* $n \geq 1$*, all* $e \in \mathcal{P}(\mathcal{M}_n)$ *and for all* $\lambda > 0$*, we have*

$$\frac{1}{\tau(e)}\tau(\mathbb{1}_{(\lambda,\infty)}(s_c((x-x_n)e))) \leq 2\exp(-\frac{c\lambda}{\|x\|_{\mathsf{bmo}^c}}),$$

with c an absolute constant. Here $\mathbb{1}_{(\lambda,\infty)}(a)$ *denotes the spectral projection of a positive operator a corresponding to the interval* (λ,∞).

Proof. By homogeneity, we can assume $\|x\|_{\mathsf{bmo}^c} = 1$. We first deal with the case $\lambda \geq 2c_1$, where c_1 is the constant in inequality (1.2.4). Let $p = \lambda/(2c_1) \geq 1$, by Chebychev's inequality and Theorem 2.16,

$$\begin{aligned}\tau(\mathbb{1}_{(\lambda,\infty)}(s_c((x-x_n)e))) &\leq \tau(e)\frac{\|(x-x_n)e\|^p_{\mathsf{h}^c_p}}{\lambda^p}\\ &\leq \tau(e)(c_1 p\lambda^{-1})^p = \tau(e)\exp(p\ln(c_1 p\lambda^{-1})) = \tau(e)\exp(-\frac{\ln 2}{2c_1}\lambda).\end{aligned}$$

When $0 < \lambda < 2c_1$,

$$\frac{1}{\tau(e)}\tau(\mathbb{1}_{(\lambda,\infty)}(s_c((x-x_n)e))) \leq 1 < 2\exp(-\frac{\ln 2}{2c_1}\lambda).$$

Therefore, we obtain the desired result by letting $c = \ln 2/(2c_1)$. □

Based on the crude version of Junge/Musat's John-Nirenberg inequality in Theorem 2.10 (resp. Corollary 2.8) for $\mathsf{bmo}(\mathcal{M})$ (resp. $\mathcal{BMO}(\mathcal{M})$), the argument in the proof of Theorem 2.16 can be adapted to get the fine version of Junge/Musat's John-Nirenberg inequality.

Corollary 2.18. *For all $0 < p < \infty$, we have*

$$\alpha_p^{-1}\|x\|_{\mathsf{bmo}} \leq \mathcal{P}\mathsf{b}_p(x) \leq \beta_p\|x\|_{\mathsf{bmo}},$$

where

$$\begin{aligned}\mathcal{P}\mathsf{b}_p(x) = \max\{\sup_n \|(dx_n)_n\|_\infty,\quad &\sup_n \sup_{e\in\mathcal{M}_n} \|(x-x_n)\frac{e}{(\tau(e))^{1/p}}\|_p,\\ &\sup_n \sup_{e\in\mathcal{M}_n} \|\frac{e}{(\tau(e))^{1/p}}(x-x_n)\|_p\}.\end{aligned}$$

The constants α_p and β_p have the same orders as those in Theorem 2.3.

Corollary 2.19. *For $0 < p < \infty$, we have*

$$\alpha_p^{-1}\|x\|_{\mathcal{BMO}} \leq \mathcal{PB}_p(x) \leq \beta_p\|x\|_{\mathcal{BMO}},$$

where

$$\begin{aligned}\mathcal{PB}_p(x) = \max\{&\sup_n \sup_{e\in\mathcal{M}_n} \|(x-x_{n-1})\frac{e}{(\tau(e))^{1/p}}\|_p,\\ &\sup_n \sup_{e\in\mathcal{M}_n} \|\frac{e}{(\tau(e))^{1/p}}(x-x_{n-1})\|_p\}.\end{aligned}$$

The constant α_p and β_p have the same orders as those in Theorem 2.3.

Again, based on Corollary 2.19, by arguments similar to the proof of Thoerem 2.17, we obtain the exponential integrability form of the John-Nirenberg inequality for $\mathcal{BMO}(\mathcal{M})$.

Theorem 2.20. *Let $x \in \mathcal{BMO}(\mathcal{M})$. Then for all natural numbers $n \geq 1$, all $e \in \mathcal{P}(\mathcal{M}_n)$ and for all $\lambda > 0$, we have*

$$\frac{1}{\tau(e)}\tau(\mathbb{1}_{(\lambda,\infty)}(|(x-x_{n-1})e|) + \mathbb{1}_{(\lambda,\infty)}(|e(x-x_{n-1})|)) \leq 4\exp(-\frac{c\lambda}{\|x\|_{\mathcal{BMO}}})$$

with c an absolute constant.

1.3 Atomic decomposition

1.3.1 A crude version of atoms

According to the crude version of the noncommutative John-Nirenberg inequality, we introduce the following

Definition 3.1. For $1 < q \leq \infty$, $a \in L_1(\mathcal{M})$ is said to be a $(1, q, c)$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and a factorization $a = yb$ such that

(i) $\mathcal{E}_n(y) = 0$;

(ii) $b \in L_{q'}(\mathcal{M}_n)$ and $\|b\|_{q'} \leq 1$;

(iii) $\|y\|_{\mathsf{h}_q^c} \leq 1$ for $1 < q < \infty$; $\|y\|_{\mathsf{bmo}^c} \leq 1$ for $q = \infty$.

Similarly, we define the notion of a $(1, q, r)$-atom with $a = yb$ replaced by $a = by$.

Lemma 3.2. *Let $1 < q \leq \infty$. If a is a $(1, q, c)$-atom, then*

$$\|a\|_{\mathsf{h}_1^c} \leq 1.$$

The analogous inequality holds for $(1, q, r)$-atoms.

Proof. We first deal with the case $1 < q < \infty$. By definition, there exists an n such that the $(1, q, c)$-atom a admits a factorization $a = yb$ as in Definition 3.1. Then

$$s_c^2(a) = b^* \sum_{k>n} \mathcal{E}_{k-1}|dy_k|^2 b = b^* s_c^2(y) b.$$

Thus by Hölder's inequality,

$$\|a\|_{\mathsf{h}_1^c} = \|s_c(a)\|_1 \leq \|s_c(y)\|_q \|b\|_{q'} \leq 1.$$

For the case $q = \infty$, the calculation is a bit different,

$$\begin{aligned} \|a\|_{\mathsf{h}_1^c} &= \left\|b^* s_c^2(y) b\right\|_{1/2}^{1/2} = \tau(\mathcal{E}_n(b^* s_c^2(y) b)^{1/2}) \\ &\leq \tau((\mathcal{E}_n(b^* s_c(y) b))^{1/2}) \leq \|\mathcal{E}_n(s_c(y))\|_\infty \|b\|_1 \\ &\leq \|y\|_{\mathsf{bmo}^c} \|b\|_1 \leq 1. \end{aligned}$$

We have used the trace preserving property of conditional expectations in the fourth equality and the operator Jensen inequality in the first inequality. For the second inequality, we have used the property that $\mathcal{E}_n \cdot \mathcal{E}_{k-1} = \mathcal{E}_n$ for all $k > n$ and Hölder's inequality. $\square$

Definition 3.3. We define $\mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M})$ as the Banach space of all $x \in L_1(\mathcal{M})$ which admit a decomposition $x = \sum_k \lambda_k a_k$, where for each k, a_k a $(1, q, c)$-atom or an element in the unit ball of $L_1(\mathcal{M}_1)$, and $\lambda_k \in \mathbb{C}$ satisfying $\sum_k |\lambda_k| < \infty$. We equip this space with the norm

$$\|x\|_{\mathsf{h}_{1,\mathrm{at}_q}^c} = \inf \sum_k |\lambda_k|,$$

where the infimum is taken over all decompositions of x described above. Similarly, we define $\mathsf{h}_{1,\mathrm{at}_q}^r(\mathcal{M})$.

Now, by Lemma 3.2, we have the obvious inclusion $\mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M}) \subset \mathsf{h}_1^c(\mathcal{M})$. In fact, the two spaces coincide thanks to the following theorem.

Theorem 3.4. *For all $1 < q \leq \infty$, we have*

$$\mathsf{h}_1^c(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M})$$

with equivalent norms. Similarly, $\mathsf{h}_1^r(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at}_q}^r(\mathcal{M})$ with equivalent norms.

We prove this theorem by duality. We require the following lemmas.

Lemma 3.5. (i) *For all $1 < q \leq 2$, $L_2(\mathcal{M})$ densely and continuously embeds into $\mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M})$.*

(ii) *For all $2 < q \leq \infty$, $L_q(\mathcal{M})$ densely and continuously embeds into $\mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M})$.*

Proof. (i). For any $x \in L_2(\mathcal{M})$, we decompose it as a linear combination of two atoms:

$$x = \|x - \mathcal{E}_1(x)\|_2 \frac{x - \mathcal{E}_1(x)}{\|x - \mathcal{E}_1(x)\|_2} + \|\mathcal{E}_1(x)\|_2 \frac{\mathcal{E}_1(x)}{\|\mathcal{E}_1(x)\|_2}.$$

Indeed, on the one hand, $\mathcal{E}_1(x)/\|\mathcal{E}_1(x)\|_2 \in L_2(\mathcal{M}_1) \subset L_1(\mathcal{M}_1)$ and

$$\|\frac{\mathcal{E}_1(x)}{\|\mathcal{E}_1(x)\|_2}\|_1 = \frac{\|\mathcal{E}_1(x)\|_1}{\|\mathcal{E}_1(x)\|_2} \leq 1.$$

On the other hand,

$$\frac{x - \mathcal{E}_1(x)}{\|x - \mathcal{E}_1(x)\|_2} = \frac{x - \mathcal{E}_1(x)}{\|x - \mathcal{E}_1(x)\|_2} \cdot \mathbb{1} \doteq y \cdot b.$$

Clearly, $\mathcal{E}_1(y) = 0$, $\|b\|_{q'} \leq 1$ and

$$\|y\|_{\mathsf{h}_q^c} = \|\frac{x - \mathcal{E}_1(x)}{\|x - \mathcal{E}_1(x)\|_2}\|_{\mathsf{h}_q^c} \leq \|\frac{x - \mathcal{E}_1(x)}{\|x - \mathcal{E}_1(x)\|_2}\|_{\mathsf{h}_2^c} \leq 1.$$

Thus x is a sum of two atoms and

$$\|x\|_{\mathsf{h}_{1,\mathrm{at}_q}^c} \leq \|x - \mathcal{E}_1(x)\|_2 + \|\mathcal{E}_1(x)\|_2 \leq \sqrt{2}\|x\|_2.$$

The density is trivial.

(ii). This case is similar to the previous one. We first deal with the case $2 < q < \infty$. Given $x \in L_q(\mathcal{M})$, we write again:

$$x = c_q\|x - \mathcal{E}_1(x)\|_q \frac{x - \mathcal{E}_1(x)}{c_q\|x - \mathcal{E}_1(x)\|_q} + \|\mathcal{E}_1(x)\|_q \frac{\mathcal{E}_1(x)}{\|\mathcal{E}_1(x)\|_q},$$

where c_q is fixed below. Indeed, $\mathcal{E}_1(x)/\|\mathcal{E}_1(x)\|_q \in L_q(\mathcal{M}_1) \subset L_1(\mathcal{M}_1)$ and

$$\|\frac{\mathcal{E}_1(x)}{\|\mathcal{E}_1(x)\|_q}\|_1 = \frac{\|\mathcal{E}_1(x)\|_1}{\|\mathcal{E}_1(x)\|_q} \leq 1.$$

On the other hand,

$$\frac{x - \mathcal{E}_1(x)}{c_q\|x - \mathcal{E}_1(x)\|_q} = \frac{x - \mathcal{E}_1(x)}{c_q\|x - \mathcal{E}_1(x)\|_q} \cdot \mathbb{1} \doteq y \cdot b,$$

$$\mathcal{E}_1(\frac{x - \mathcal{E}_1(x)}{c_q\|x - \mathcal{E}_1(x)\|_q}) = 0, \quad \|b\|_{q'} \leq 1$$

and the noncommutative Burkholder inequality in [25] yields

$$\|y\|_{\mathsf{h}_q^c} = \|\frac{x - \mathcal{E}_1(x)}{c_q\|x - \mathcal{E}_1(x)\|_q}\|_{\mathsf{h}_q^c} \leq c_q\|\frac{x - \mathcal{E}_1(x)}{c_q\|x - \mathcal{E}_1(x)\|_q}\|_q \leq 1.$$

Therefore,

$$\|x\|_{\mathsf{h}_{1,\mathrm{at}_q}^c} \leq c_q\|x - \mathcal{E}_1(x)\|_q + \|\mathcal{E}_1(x)\|_q \leq (2c_q + 1)\|x\|_q.$$

The case $q = \infty$ is proved in the same way just by replacing the noncommutative Burkholder inequality by the trivial fact that $\|\cdot\|_{\mathsf{bmo}^c} \leq \|\cdot\|_\infty$. The density is trivial. $\square$

Lemma 3.6. *Let $1 < q < \infty$. Then*

$$(\mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M}))^* = \mathsf{bmo}_{q'}^c(\mathcal{M})$$

with equivalent norms. More precisely,

(i) *Every $x \in \mathsf{bmo}_{q'}^c(\mathcal{M})$ defines a bounded linear functional on $\mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M})$ by*

$$\varphi_x(a) = \tau(x^*a), \forall a \in (1, q, c)\text{-atoms}. \tag{1.3.1}$$

(ii) *Conversely, each $\varphi \in (\mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M}))^*$ is given as (1.3.1) by some $x \in \mathsf{bmo}_{q'}^c(\mathcal{M})$.*

Similarly, $(\mathsf{h}_{1,\mathrm{at}_q}^r(\mathcal{M}))^* = \mathsf{bmo}_{q'}^r(\mathcal{M})$ with equivalent norms.

Proof. (i) Let $x \in \mathsf{bmo}_{q'}^c$, and $a = yb$ where a is a $(1, q, c)$-atom as in Definition 3.1. Then

$$\begin{aligned}|\tau(x^*a)| &= |\tau(\mathcal{E}_n(x^*y)b)| \\ &= |\tau(\mathcal{E}_n((x^* - x_n^*)y)b)| = |\tau(((x - x_n)b^*)^*y)|.\end{aligned}$$

Thus, by the duality identity $\mathsf{h}_q^c(\mathcal{M}) = (\mathsf{h}_{q'}^c(\mathcal{M}))^*$ (see [25] for the relevant constants),

$$|\tau(x^*a)| \leq \|(x - x_n)b^*\|_{\mathsf{h}_{q'}^c} \|y\|_{\mathsf{h}_q^c} \leq \|x\|_{\mathsf{bmo}_{q'}^c}.$$

(ii). Let φ be any linear functional on $\mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M})$. When $1 < q \leq 2$, by Lemma 3.5 we can find $x \in L_2(\mathcal{M})$ such that

$$\varphi(y) = \tau(x^*y), \qquad \forall y \in L_2(\mathcal{M}),$$

and

$$\|\varphi\| = \sup_{y \in L_2, \|y\|_{\mathsf{h}_{1,\mathrm{at}_q}^c} \leq 1} |\tau(x^*y)|.$$

When $2 < q < \infty$, by the same Lemma 3.5, we get the same representation of φ with an $x \in L_{q'}(\mathcal{M})$. Then fix n and take any $b \in \mathcal{M}_n$ with $\|b\|_{q'} \leq 1$. Again, by the duality $\mathsf{h}_q^c(\mathcal{M}) = (\mathsf{h}_{q'}^c(\mathcal{M}))^*$, we do the following calculation:

$$\begin{aligned}\|(x - x_n)b\|_{\mathsf{h}_{q'}^c} &= \sup_{\|y\|_{(\mathsf{h}_{q'}^c)^*} \leq 1} |\tau(b^*(x^* - x_n^*)y)| \\ &\leq \sup_{\|y\|_{\mathsf{h}_q^c} \leq cq} |\tau(b^*(x^* - x_n^*)y)| \\ &= \sup_{\|y\|_{\mathsf{h}_q^c} \leq cq} |\tau((x^* - x_n^*)(y - y_n)b^*)|\end{aligned}$$

$$= \sup_{\|y\|_{\mathsf{h}_q^c} \leq cq} |\tau(x^*((y-y_n)b^*))|$$
$$\leq cq\|\varphi\|$$

Here, we have used the fact that $\tau(x-x_n) = \tau(y-y_n) = 0$ in the second and third equality respectively. The second inequality is due to the fact that $(y-y_n)b^*$ is a $(1,q,c)$-atom. $\square$

Now we are at a position to prove Theorem 3.4.

Proof. We consider here only the case $1 < q < \infty$ and postpone the case $q = \infty$ to the end of the proof of Theorem 3.12 below. We only need to show the inclusion

$$\mathsf{h}_1^c(\mathcal{M}) \subset \mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M}).$$

Take $x \in \mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M})$, by Theorem 2.3 and Lemma 3.6, we can conduct the following calculation,

$$\begin{aligned} \|x\|_{\mathsf{h}_{1,\mathrm{at}_q}^c} &= \sup_{\|y\|_{(\mathsf{h}_{1,\mathrm{at}_q}^c)^*} \leq 1} |\tau(x^*y)| \\ &\leq \sup_{\|y\|_{\mathsf{bmo}_{q'}^c} \leq cq} |\tau(x^*y)| \\ &\leq \sup_{\|y\|_{\mathsf{bmo}^c} \leq cq} |\tau(x^*y)| \leq cq\|x\|_{\mathsf{h}_1^c}. \end{aligned}$$

Then we end the proof with the density of $\mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M})$ in $\mathsf{h}_1^c(\mathcal{M})$. $\square$

Definition 3.7. We define

$$\mathsf{h}_{1,\mathrm{at}_q}(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at}_q}^c(\mathcal{M}) + \mathsf{h}_{1,\mathrm{at}_q}^r(\mathcal{M}) + \mathsf{h}_1^d(\mathcal{M})$$

equipped with the sum norm

$$\|x\|_{\mathsf{h}_{1,\mathrm{at}_q}} = \inf_{x = x_c + x_r + x_d} \{\|x_c\|_{\mathsf{h}_{1,\mathrm{at}_q}^c} + \|x_r\|_{\mathsf{h}_{1,\mathrm{at}_q}^r} + \|x_d\|_{\mathsf{h}_1^d}\}.$$

Then by Theorem 3.4, we obtain the atomic decomposition of $\mathsf{h}_1(\mathcal{M})$.

Corollary 3.8. *We have*

$$\mathsf{h}_1(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at}_q}(\mathcal{M})$$

with equivalent norms.

Combined with Davis' decomposition presented in [45], the above theorem yields $\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at}_q}(\mathcal{M})$ with equivalent norms. In other words, we obtain an atomic decomposition for $\mathcal{H}_1(\mathcal{M})$ too.

1.3.2 A fine version of atoms

Definition 3.9. For $1 < q \leq \infty$, $a \in L_1(\mathcal{M})$ is said to be a $(1,q,c)_{\mathrm{pr}}$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and a projection $e \in \mathcal{P}(\mathcal{M}_n)$ such that

(i) $\mathcal{E}_n(a) = 0$;

(ii) $r(a) \leq e$;

(iii) $\|a\|_{\mathsf{h}_q^c} \leq (\tau(e))^{-\frac{1}{q'}}$ for $1 < q < \infty$; $\|a\|_{\mathsf{bmo}^c} \leq (\tau(e))^{-1}$ for $q = \infty$.

Similarly, we define $(1,q,r)_{\rm pr}$-atoms with $r(a)$ replaced by $l(a)$.

Remark 3.10. A $(1,q,c)_{\rm pr}$-atom a is necessarily a $(1,q,c)$-atom. Indeed, we can factorize a as $a = yb$ with $y = a(\tau(e))^{1/q'}$ and $b = e(\tau(e))^{-1/q'}$.

Definition 3.11. We define $\mathsf{h}^c_{1,\mathrm{at}_{q,\mathrm{pr}}}(\mathcal{M})$ to be the Banach space of all $x \in L_1(\mathcal{M})$ which admit a decomposition $x = \sum_k \lambda_k a_k$, where for each k, a_k is a $(1,q,c)_{\rm pr}$-atom or an element in the unit ball of $L_1(\mathcal{M}_1)$, and $\lambda_k \in \mathbb{C}$ satisfying $\sum_k |\lambda_k| < \infty$. We equip this space with the norm

$$\|x\|_{\mathsf{h}^c_{1,\mathrm{at}_{q,\mathrm{pr}}}} = \inf \sum_k |\lambda_k|,$$

where the infimum is taken over all decompositions of x described above. Similarly, we define $\mathsf{h}^r_{1,\mathrm{at}_{q,\mathrm{pr}}}(\mathcal{M})$.

Now, by Remark 3.10 and Lemma 3.4, we have the obvious inclusion $\mathsf{h}^c_{1,\mathrm{at}_{q,\mathrm{pr}}}(\mathcal{M}) \subset \mathsf{h}^c_1(\mathcal{M})$. In fact, the two spaces coincide thanks to the following theorem.

Theorem 3.12. *For all $1 < q \le \infty$, we have*

$$\mathsf{h}^c_1(\mathcal{M}) = \mathsf{h}^c_{1,\mathrm{at}_{q,\mathrm{pr}}}(\mathcal{M})$$

with equivalent norms. Similarly, $\mathsf{h}^r_1(\mathcal{M}) = \mathsf{h}^r_{1,\mathrm{at}_{q,\mathrm{pr}}}(\mathcal{M})$ with equivalent norms.

Again, we prove this theorem for $1 < q < \infty$ by showing $(\mathsf{h}^c_{1,\mathrm{at}_{q,\mathrm{pr}}}(\mathcal{M}))^* = \mathsf{bmo}^c_{q',\mathrm{pr}}(\mathcal{M})$. The latter duality equality is proved in the same way as Theorem 3.6. We leave the details to the reader. However by the argument in Theorem 4.6, we can not prove the theorem in the case $q = \infty$, due to the lack of Riesz representation. Here we provide another way to do it, which seems new, even in the commutative case.

Let $\mathcal{P}$ be the set of projections of $\mathcal{M}$. Given $e \in \mathcal{P}$ let

$$n_e = \min\{k \ : \ e \in \mathcal{P}(\mathcal{M}_k)\}.$$

Note that $n_e = \infty$ if the set on the right hand side is empty. This case is of no interest in the discussion below. For a family $(g_e)_{e\in\mathcal{P}} \subset \mathsf{bmo}^c(\mathcal{M})$ define

$$\|(g_e)_e\|_{L_1^{\mathcal{P}}(\mathsf{bmo}^c)} = \sum_{e\in\mathcal{P}} \tau(e)\|g_e\|_{\mathsf{bmo}^c}.$$

We will consider the Banach space:

$$L_1^{\mathcal{P}}(\mathsf{bmo}^c) = \{(g_e)_e \ : \ g_e e = g_e,\ \mathcal{E}_{n_e} g_e = 0,\ \|(g_e)_e\|_{L_1^{\mathcal{P}}(\mathsf{bmo}^c)} < \infty\}.$$

We will also need the following space consisting of families in $\mathsf{h}^c_1(\mathcal{M})$:

$$L_\infty^{\mathcal{P}}(\mathsf{h}^c_1) = \{(f_e)_e \ : \ f_e e = f_e,\ \mathcal{E}_{n_e} f_e = 0,\ \|(f_e)_e\|_{L_\infty^{\mathcal{P}}(\mathsf{h}^c_1)} < \infty\},$$

where

$$\|(f_e)_e\|_{L_\infty^{\mathcal{P}}(\mathsf{h}^c_1)} = \sup_{e\in\mathcal{P}} \frac{1}{\tau(e)}\, \|f_e\|_{\mathsf{h}^c_1}.$$

For convenience, we denote $L_1^{\mathcal{P}}(\mathsf{bmo}^c)$ by X and $L_\infty^{\mathcal{P}}(\mathsf{h}^c_1)$ by Z. We embed $\mathsf{bmo}^c_{1,\mathrm{pr}}(\mathcal{M})$ isomorphically into Z via the following map

$$\pi(y) = ((y - y_{n_e})e)_e.$$

Set $Y = \pi(\mathsf{bmo}^c_{1,\mathrm{pr}}(\mathcal{M}))$.

Lemma 3.13. *With the notation above we have*

(i) *Z is a subspace of X^* with equivalent norms, so is Y.*

(ii) *Y is w^*-closed in X^*.*

Proof. (i). Let $(f_e)_e \in Z$, for any $(g_e)_e \in X$, we have

$$\begin{aligned}|\langle (f_e)_e, (g_e)_e\rangle| &= |\sum_e \tau((f_e)^* g_e)|\\ &\leq \sqrt{2}\sum_e \|f_e\|_{\mathsf{h}_1^c}\|g_e\|_{\mathsf{bmo}^c}\\ &\leq \sqrt{2}\sup_e \frac{1}{\tau(e)}\|f_e\|_{\mathsf{h}_1^c}\cdot \sum_e \tau(e)\|g_e\|_{\mathsf{bmo}^c}\\ &= \sqrt{2}\|(f_e)_e\|_Z\|(g_e)_e\|_X.\end{aligned}$$

Thus we get $\|(f_e)_e\|_{X^*} \leq \sqrt{2}\|(f_e)_e\|_Z$.

We turn to the proof of the inverse inequality. For any $(f_e)_e \in Z$, fix $e_0 \in \mathcal{P}$, we have

$$\begin{aligned}\frac{1}{\tau(e_0)}\|f_{e_0}\|_{\mathsf{h}_1^c} &= \sup_{\|g\|_{\mathsf{bmo}^c}\leq 1}\frac{1}{\tau(e_0)}\left|\tau((f_{e_0})^* g)\right|\\ &= \sup_{\|g\|_{\mathsf{bmo}^c}\leq 1}\frac{1}{\tau(e_0)}\left|\tau((f_{e_0})^*(g-g_{n_{e_0}})e_0)\right|\\ &\leq \sup_{\|(g-g_{n_{e_0}})e_0\|_{\mathsf{bmo}^c}\leq 1}\frac{1}{\tau(e_0)}\left|\tau((f_{e_0})^*(g-g_{n_{e_0}})e_0)\right|.\end{aligned}$$

Then we define $(g_e)_e$ as $g_e = (g-g_{n_{e_0}}e_0)/\tau(e_0)$ if $e = e_0$, otherwise $g_e = 0$. Thus

$$\frac{1}{\tau(e_0)}\|f_{e_0}\|_{\mathsf{h}_1^c} \leq \|(f_e)_e\|_{X^*}\|(g_e)_e\|_X \leq \|(f_e)_e\|_{X^*},$$

which implies $\|(f_e)_e\|_Z \leq \|(f_e)_e\|_{X^*}$.

(ii). Since Y is a subspace of X^*, by Krein and Smulian's theorem, we only need to prove that for all $t > 0$, $Y \cap B_t(X^*)$ is w*-closed in X^*, where $B_t(X^*)$ is the closed ball of X^* centered at the origin and with radius t. Take a net $(y^\alpha)_\alpha \subset \mathsf{bmo}^c_{1,\mathrm{pr}}(\mathcal{M})$ such that $\pi((y^\alpha)_\alpha) \subset Y \cap B_t(X^*)$. Hence $(y^\alpha)_\alpha$ are bounded in $\mathsf{bmo}^c_{1,\mathrm{pr}}(\mathcal{M})$. Suppose that,

$$\langle \pi(y^\alpha), (g_e)_e\rangle \to \langle \xi, (g_e)_e\rangle, \qquad \forall (g_e)_e \in X, \tag{1.3.2}$$

for some $\xi \in B_t(X^*)$. We will show that $\xi \in Y$, which will complete the proof. We need two facts. The first one is that $\mathsf{bmo}^c_{1,\mathrm{pr}}(\mathcal{M})$ is a dual space by Theorem 2.16, so its unit ball is w*-compact. Therefore, the bounded net $(y^\alpha)_\alpha$ in $\mathsf{bmo}^c_{1,\mathrm{pr}}(\mathcal{M})$ admits a w^*-cluster point y. Without loss of generality, we assume that $(y^\alpha)_\alpha$ converges to y in the w^*-topology:

$$\langle y^\alpha, x\rangle \to \langle y, x\rangle, \qquad \forall x \in \mathsf{h}_1^c(\mathcal{M}). \tag{1.3.3}$$

The second fact is that for any $(g_e)_e \in X$, the sum $\sum_e g_e$ is absolutely summable in $\mathsf{h}_1^c(\mathcal{M})$. Indeed, by Lemma 3.2

$$\sum_e \|g_e\|_{\mathsf{h}_1^c} \leq \sum_e \tau(e)\|g_e\|_{\mathsf{bmo}^c} = \|(g_e)_e\|_X.$$

Therefore, for any $(g_e)_e \in X$, we have

$$\begin{aligned}\langle \pi(y^\alpha), (g_e)_e \rangle &= \sum_e \tau(((y_e^\alpha - y_{n_e}^\alpha)e)^* g_e) \\ &= \tau((y^\alpha)^* \sum_e g_e)\end{aligned}$$

Combining 1.3.2 and 1.3.3, we deduce that $\xi = \pi(y) \in Y$, as desired. □

We can now prove Theorem 3.12 in the case of $q = \infty$.
Proof. Let $Y_\perp$ be the preannihilator of Y in X^*:

$$Y_\perp = \{(g_e)_e \in X \ : \ \langle \pi(y), (g_e)_e \rangle = 0, \forall y \in \mathsf{bmo}_{1,\mathrm{pr}}^c(\mathcal{M})\}.$$

Then by the bipolar theorem

$$Y \simeq (X/Y_\perp)^*.$$

Using the second fact in the proof of the previous lemma, we get

$$\begin{aligned}Y_\perp &= \{(g_e)_e \in X \ : \ \tau(y^* \sum_e g_e) = 0, \forall y \in \mathsf{bmo}_{1,\mathrm{pr}}^c(\mathcal{M})\} \\ &= \{(g_e)_e \in X \ : \ \sum_e g_e = 0 \text{ in } \mathsf{h}_1^c(\mathcal{M})\}.\end{aligned}$$

Then for $(g_e)_e \in X/Y_\perp$, let

$$g = \sum_{e \in \mathcal{P}} g_e.$$

Then

$$\begin{aligned}\|(g_e)_e\|_{X/Y_\perp} &= \inf\{\sum_e \tau(e)\, \|(g'_e)_e\|_{\mathsf{bmo}^c} \ : \ g = \sum_e g'_e,\ (g'_e)_e \in X\} \\ &= \inf\{\sum_e |\lambda_e| \ : \ g = \sum_e \lambda_e a_e,\ (\lambda_e a_e)_e \in X,\ \|a_e\|_{\mathsf{bmo}^c} \leq \frac{1}{\tau(e)}\} \\ &= \|g\|_{\mathsf{h}_{1,\mathrm{at}_\infty,\mathrm{pr}}^c}.\end{aligned}$$

Consequently, for any $x \in \mathsf{h}_{1,\mathrm{at}_\infty,\mathrm{pr}}^c(\mathcal{M})$ and any decomposition $x = \sum_e \lambda_e a_e$,

$$\begin{aligned}\|x\|_{\mathsf{h}_{1,\mathrm{at}_\infty,\mathrm{pr}}^c} &= \|(\lambda_e a_e)_e\|_{X/Y_\perp} \\ &= \|(\lambda_e a_e)_e\|_{Y^*} \\ &= \sup_{y \in \mathsf{bmo}_{1,\mathrm{pr}}^c, \|\pi(y)\|_Y \leq 1} |\langle (\lambda_e a_e), \pi(y) \rangle| \\ &\leq \sup_{\|y\|_{\mathsf{bmo}^c} \leq c} |\tau((\sum_e \lambda_e a_e)^* y)| \leq c \|x\|_{\mathsf{h}_1^c}.\end{aligned}$$

Therefore, combined with Lemma 3.2 and Remark 3.10, the density of $\mathsf{h}_{1,\mathrm{at}_\infty,\mathrm{pr}}^c(\mathcal{M})$ in $\mathsf{h}_1^c(\mathcal{M})$ (due to Lemma 3.5) yields the desired duality identity $\mathsf{h}_{1,\mathrm{at}_\infty,\mathrm{pr}}^c(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M})$. □

Let us return back to the unsettled case $q = \infty$ in the proof of Theorem 3.4. Since a fine atom is necessarily a crude atom, we get $\mathsf{h}_1^c(\mathcal{M}) \subset \mathsf{h}_{1,\mathrm{at}_\infty}^c(\mathcal{M})$, hence $\mathsf{h}_1^c(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at}_\infty}^c(\mathcal{M})$ with equivalent norms due to Lemma 3.2. Thus Theorem 3.4 is completely proved.

Definition 3.14. We define

$$\mathsf{h}_{1,\mathrm{at}_{q,\mathrm{pr}}}(\mathcal{M}) = \mathsf{h}^c_{1,\mathrm{at}_{q,\mathrm{pr}}}(\mathcal{M}) + \mathsf{h}^r_{1,\mathrm{at}_{q,\mathrm{pr}}}(\mathcal{M}) + \mathsf{h}^d_1(\mathcal{M})$$

equipped with the sum norm

$$\|x\|_{\mathsf{h}_{1,\mathrm{at}_{q,\mathrm{pr}}}} = \inf_{x=x_c+x_r+x_d}\{\|x_c\|_{\mathsf{h}^c_{1,\mathrm{at}_{q,\mathrm{pr}}}} + \|x_r\|_{\mathsf{h}^r_{1,\mathrm{at}_{q,\mathrm{pr}}}} + \|x_d\|_{\mathsf{h}^d_1}\}.$$

Then by Theorem 3.12 and Perrin's noncommutative Davis decomposition (see [45]), we get the atomic decomposition of $\mathsf{h}_1(\mathcal{M})$ and $\mathcal{H}_1(\mathcal{M})$.

Corollary 3.15. *We have*

$$\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1(\mathcal{M}) = \mathsf{h}_{1,\mathrm{at}_{q,\mathrm{pr}}}(\mathcal{M}),$$

for any $1 < q \leq \infty$, *with equivalent norms.*

However, using Corollary 2.18, we can obtain another kind of atomic decomposition for $\mathsf{h}_1(\mathcal{M})$ or $\mathcal{H}_1(\mathcal{M})$, which is exactly the noncommutative analogue of the classical case.

Definition 3.16. For $1 < q \leq \infty$, $a \in L_1(\mathcal{M})$ is said to be a $(1,q)$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and a projection $e \in \mathcal{P}(\mathcal{M}_n)$ such that
(i) $\mathcal{E}_n(a) = 0$;
(ii) $r(a) \leq e$ or $l(a) \leq e$;
(iii) $\|a\|_q \leq (\tau(e))^{-\frac{1}{q'}}$ for $1 < q \leq \infty$.

Definition 3.17. We define $\mathsf{h}^{\mathrm{at}}_{1,q}(\mathcal{M})$ as the Banach space of all $x \in L_1(\mathcal{M})$ which admit a decomposition $x = y + \sum_k \lambda_k a_k$, where for each k, a_k is a $(1,q)$-atom or an element in the unit ball of $L_1(\mathcal{M}_1)$, $\lambda_k \in \mathbb{C}$ satisfying $\sum_k |\lambda_k| < \infty$, and where the martingale differences of y satisfy $\sum_{j\geq 1}\|dy_j\|_1 < \infty$. We equip this space with the norm

$$\|x\|_{\mathsf{h}^{\mathrm{at}}_{1,q}} = \inf\big\{\sum_j \|dy_j\|_1 + \sum_k |\lambda_k|\big\},$$

where the infimum is taken over all decompositions of x as above.

Lemma 3.18. *If a is a $(1,q)$-atom, then*

$$\|a\|_{\mathsf{h}_1} \leq \frac{cq}{q-1}.$$

Proof. Without loss of generality, suppose a is a $(1,q)$-atom with $r(a) \leq e$. We apply Corollary 3.18 and the duality $(\mathsf{h}_1(\mathcal{M}))^* = \mathsf{bmo}(\mathcal{M})$.

$$\begin{aligned}\|a\|_{\mathsf{h}_1} &\leq c \sup_{\|x\|_{\mathsf{bmo}}\leq 1} \tau(x^*a)\\ &= c \sup_{\|x\|_{\mathsf{bmo}}\leq 1} \tau((x-x_n)^*a)\\ &= c \sup_{\|x\|_{\mathsf{bmo}}\leq 1} \tau(((x-x_n)e)^*a)\\ &\leq c\|a\|_q\|(x-x_n)e\|_{q'} \leq cq'.\end{aligned}$$

□

Theorem 3.19. *For all* $1 < q \leq \infty$, *we have*

$$\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1(\mathcal{M}) = \mathsf{h}^{\mathrm{at}}_{1,q}(\mathcal{M})$$

with equivalent norms.

By Lemma 3.18, Corollary 2.18 and using arguments similar to those in the proof of Theorem 3.4, we can prove the theorem for the case $1 < q < \infty$. For the case $q = \infty$, we use the argument in Theorem 3.12. Instead of $L_1^{\mathcal{P}}(\mathsf{bmo}^c)$ and $L_\infty^{\mathcal{P}}(\mathsf{h}_1^c)$, we consider the following two spaces:

$$L_1^{\mathcal{P}}(L_\infty) = \big\{(g_e)_e \ : \ g_e e = g_e \,\mathrm{or}\, e g_e = g_e, \mathcal{E}_{n_e} g_e = 0, \ \|(g_e)_e\|_{L_1^{\mathcal{P}}(L_\infty)} < \infty\big\},$$
$$L_\infty^{\mathcal{P}}(L_1) = \big\{(f_e)_e \ : \ f_e e = f_e \,\mathrm{or}\, e f_e = f_e, \mathcal{E}_{n_e} f_e = 0, \ \|(f_e)_e\|_{L_\infty^{\mathcal{P}}(L_1)} < \infty\big\},$$

where

$$\|(g_e)_e\|_{L_1^{\mathcal{P}}(L_\infty)} = \sum_e \tau(e)\, \|g_e\|_\infty,$$
$$\|(f_e)_e\|_{L_\infty^{\mathcal{P}}(L_1)} = \max\big\{ \sup_e \frac{1}{\tau(e)}\, \|f_e e\|_1, \ \sup_e \frac{1}{\tau(e)}\, \|e f_e\|_1\big\}.$$

Then by Lemma 3.18 and Corollary 2.18, we get the announced results. We leave the details to the reader.

Remark 3.20. The part of this paper on the crude versions of the John-Nirenberg inequalities and atomic decomposition can be easily extended to the type III case with minor modifications.

1.4 An open question of Junge and Musat

It is an open question asked in [23] (on page 136) that given $2 < p < \infty$, whether there exists a constant c_p such that

$$\sup_k \|\mathcal{E}_k|x - \mathcal{E}_{k-1}x|^p\|_\infty^{\frac{1}{p}} \leq c_p \|x\|_{\mathcal{BMO}}? \tag{1.4.1}$$

It is easy to see that the answer is negative for matrix-valued functions with irregular filtration. In the following, we show that the answer is negative even for matrix-valued dyadic martingales. Recall that Remark 2.14 already shows that the answer is negative if one considers the column norm $\|\cdot\|_{\mathcal{BMO}^c}$ alone on the right hand side.

Let $\mathcal{M}$ and $\mathcal{M}_k$ be as in Remark 2.14. We consider this special case and show that the best constant $c_p(n)$ such that (1.4.1) holds is bigger than $c(\log(n+1))^{1/p}$ for all $p \geq 3$. Let b be an M_n-valued function on $\mathbb{T}$. We need the so-called "sweep" function of b

$$S(b) = \sum_{k=1}^\infty |db_k|^2.$$

Note that it is just the square of the usual square function. Matrix-valued sweep functions have been studied in [4], [11], [33] etc. It is proved in [33] that the best constant c_n such that

$$\|S(b)\|_{\mathcal{BMO}^c} \leq c_n \|b\|_\infty^2 \tag{1.4.2}$$

is $c(\log(n+1))^2$. A similar result had been proved previously by Blasco and Pott (see [4]) by considering $\|b\|^2_{\mathcal{BMO}^c}$ on the right side of (1.4.2).

Lemma 4.1. *Assume* $\|f\|_{\mathcal{BMO}^c} \leq c(n)\sup_k \|\mathcal{E}_k|f-\mathcal{E}_{k-1}f|\|_\infty$ *for any selfadjoint* f. *Then* $c(n) \geq c(\log(n+1))^2$.

Proof. Under the assumption, we have

$$\begin{aligned}
\|S(b)\|_{\mathcal{BMO}^c} &\leq c(n)\sup_m \|\mathcal{E}_m|S(b)-\mathcal{E}_{m-1}S(b)|dt\|_\infty \\
&= c(n)\sup_m \Big\|\mathcal{E}_m\Big|\sum_{k=1}^\infty |db_k|^2 - \mathcal{E}_{m-1}\sum_{k=1}^\infty |db_k|^2\Big|\Big\|_\infty \\
&= c(n)\sup_m \Big\|\mathcal{E}_m\Big|\sum_{k=m}^\infty |db_k|^2 - \mathcal{E}_{m-1}\sum_{k=m}^\infty |db_k|^2\Big|\Big\|_\infty.
\end{aligned}$$

Let $x = \sum_{k=m}^\infty |db_k|^2$ and $y = \mathcal{E}_{m-1}\sum_{k=m}^\infty |db_k|^2$. By the convexity of $|\cdot|^2$, we get

$$|\frac{x-y}{2}|^2 \leq \frac{|x|^2+|y|^2}{2} \leq \frac{|x|^2+\|y\|_\infty^2\mathbb{1}}{2} \leq \frac{(|x|+\|y\|_\infty\mathbb{1})^2}{2}.$$

Then by Löwner-Heinz's inequality,

$$|\frac{x-y}{2}| \leq \frac{|x|+\|y\|_\infty\mathbb{1}}{\sqrt{2}}.$$

Thus by the triangle inequality, we have

$$\begin{aligned}
\|S(b)\|_{\mathcal{BMO}^c} &\leq 2c(n)\sup_m \|\mathcal{E}_m x + \|y\|_\infty\mathbb{1}\|_\infty \\
&= 2c(n)\sup_m \|\mathcal{E}_m|b-\mathcal{E}_{m-1}b|^2\|_\infty + 2c(n)\|\mathcal{E}_{m-1}|b-\mathcal{E}_{m-1}b|^2\|_\infty \\
&\leq 2c(n)\|b\|_{\mathcal{BMO}^c}^2 + 2c(n)\|\mathcal{E}_m|b-\mathcal{E}_{m-1}b|^2\|_\infty \\
&\leq 4c(n)\|b\|_{\mathcal{BMO}^c}^2.
\end{aligned}$$

We then get $c(n) \geq c(\log(n+1))^2$ by (1.4.2). □

Lemma 4.2. *Let* $0 < p < \infty$ *and* $\mathcal{E}_m$ *be the conditional expectation from* $\mathcal{M}$ *onto* $\mathcal{M}_m$, *we have*

$$\|\mathcal{E}_m|x|^{\frac{p+1}{2}}\|_\infty \leq \|\mathcal{E}_m|x|^p\|_\infty^{\frac{1}{2}}\|\mathcal{E}_m|x|\|_\infty^{\frac{1}{2}}.$$

Proof. By Hölder's inequality, we get

$$\begin{aligned}
\|\mathcal{E}_m|x|^{\frac{p+1}{2}}\|_\infty &= \sup_{\|a\|_{L_1^+(\mathcal{M}_m)\leq 1}} \tau(\mathcal{E}_m|x|^{\frac{p+1}{2}}a) \\
&= \sup_{\|a\|_{L_1^+(\mathcal{M}_m)\leq 1}} \tau(a^{\frac{1}{2}}|x|^{\frac{p}{2}}|x|^{\frac{1}{2}}a^{\frac{1}{2}}) \\
&\leq \sup_{\|a\|_{L_1^+(\mathcal{M}_m)\leq 1}} (\tau(a|x|^p))^{\frac{1}{2}}(\tau(a|x|))^{\frac{1}{2}} \\
&= \|\mathcal{E}_m|x|^p\|_\infty^{\frac{1}{2}}\|\mathcal{E}_m|x|\|_\infty^{\frac{1}{2}}.
\end{aligned}$$

□

Theorem 4.3. *Suppose* $\sup_k \|\mathcal{E}_k|f-\mathcal{E}_{k-1}f|^p\|_\infty^{1/p} \leq c_p(n)\|f\|_{\mathcal{BMO}}$ *for some* $p \geq 3$. *Then*

$$c_p(n) \geq c(\log(n+1))^{\frac{2}{p}}.$$

Proof. Fix a selfadjoint M_n-valued function b. By the operator Jensen inequality and Lemma 4.2, for $p \geq 3$,

$$\begin{aligned}
\|b\|^2_{\mathcal{BMO}} &= \sup_m \|\mathcal{E}_m|b-\mathcal{E}_{m-1}b|^2\|_\infty \\
&\leq \sup_m \|\mathcal{E}_m|b-\mathcal{E}_{m-1}b|^{\frac{p+1}{2}}\|_\infty^{\frac{4}{p+1}} \\
&\leq \sup_m \|\mathcal{E}_m|b-\mathcal{E}_{m-1}b|^p\|_\infty^{\frac{2}{p+1}} \sup_m \|\mathcal{E}_m|b-\mathcal{E}_{m-1}b|\|_\infty^{\frac{2}{p+1}} \\
&\leq (c_p(n)\|b\|_{\mathcal{BMO}})^{\frac{2p}{p+1}} \sup_m \|\mathcal{E}_m|b-\mathcal{E}_{m-1}b|\|_\infty^{\frac{2}{p+1}}.
\end{aligned}$$

Then

$$\|b\|_{\mathcal{BMO}} \leq (c_p(n))^p \sup_m \|\mathcal{E}_m|b-\mathcal{E}_{m-1}b|\|_\infty.$$

By Lemma 4.1, we get

$$(c_p(n))^p \geq c(\log(n+1))^2.$$

□

From Theorem 4.3, we get a negative answer for the open question by letting $n \to \infty$.

Chapter 2

Wavelet approach to operator-valued Hardy spaces

Introduction

In this chapter, we exploit Meyer's wavelet methods to the study of the operator-valued Hardy spaces. We are motivated by two rapidly developed fields. The first one is the theory of noncommutative martingales inequalities. This theory had been already initiated in the 1970's. Its modern period of development has begun with Pisier and Xu's seminal paper [48] in which the authors established the noncommutative Burkholder-Gundy inequalities and Fefferman duality theorem between H_1 and BMO. Since then many classical results have been successfully transferred to the noncommutative world (see [25], [28], [34], [2]). In particular, motivated by [17], Mei [34] developed the theory of Hardy spaces on $\mathbb{R}^n$ for operator-valued functions.

Our second motivation is the theory of wavelets founded by Meyer. It is nowadays well known that this theory is important for many domains, in particular in harmonic analysis. For instance, it provides powerful tools to the theory of Calderón-Zygmund singular integral operators. More recently, Meyer's wavelet methods were extended to study more sophistical subjects in harmonic analysis. For example, the authors of [9] exploited the properties of Meyer's wavelets to give a characterization of product BMO by commutators; [39] deals with the estimates of bi-parameter paraproducts.

It is in this spirit that we wish to understand how useful wavelet methods are for noncommutative analysis. The most natural and possible way would be first to do this in the semi-commutative case. This is exactly the purpose of the present chapter which could be viewed as the first attempt towards the development of wavelet techniques for noncommutative analysis.

A wavelet basis of $L_2(\mathbb{R})$ is a complete orthonormal system $(w_I)_{I\in\mathcal{D}}$, where $\mathcal{D}$ denotes the collection of all dyadic intervals in $\mathbb{R}$, w is a Schwartz function satisfying the properties needed for Meryer's construction in [38], and

$$w_I(x) \doteq \frac{1}{|I|^{\frac{1}{2}}} w\big(\frac{x-c_I}{|I|}\big),$$

where c_I is the center of I. The central facts that we will need about the wavelet basis are the orthogonality between different w_I's, $\|w\|_{L_2(\mathbb{R})} = 1$ and the regularity of w,

$$\max(|w(x)|, |w'(x)|) \lesssim (1+|x|)^{-m}, \quad \forall m \geq 2.$$

The analogy between wavelets and dyadic martingales is well known. The key observation is the following parallelism:

$$\sum_{|I|=2^{-n+1}} \langle f, w_I \rangle w_I \sim df_n,$$

where df_n denotes the n-th dyadic martingale difference of f. As dyadic martingales are much easier to handle, this parallelism explains why wavelet approach to many problems in harmonic analysis is usually simple and efficient. On the other hand, it also indicates that martingale methods may be used to deal with wavelets. With this in mind, we develop the operator-valued Hardy spaces based on the wavelet methods in the way which is well known in the noncommutative martingales case. Then we show that our Hardy and BMO spaces coincide with Mei's. In other words, we provide another approach, which is much simpler than Mei's original one, to recover all the results of [34].

This chapter is organized as follows. In section 2.1, we will give some preliminaries on noncommutative analysis, the definition of $\mathcal{H}_p(\mathbb{R},\mathcal{M})$ with $1 \leq p < \infty$ and $L_q\mathcal{MO}(\mathbb{R},\mathcal{M})$ with $2 < q \leq \infty$ in our setting. In section 2.2, we are concerned with three duality results. The most important one is the noncommutative analogue of the famous Fefferman duality theorem between $\mathcal{H}_1^c(\mathbb{R},\mathcal{M})$ and $\mathcal{BMO}^c(\mathbb{R},\mathcal{M})$. The second one is the duality between $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ and $L_{p'}^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ with $1 < p < 2$, where we need the noncommutative Doob's inequality, this is why we consider the case $1 < p < 2$ independently. The last one is the duality between $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ and $\mathcal{H}_{p'}^c(\mathbb{R},\mathcal{M})$ with $1 < p < \infty$. As a corollary of the last two results, we identify $\mathcal{H}_q^c(\mathbb{R},\mathcal{M})$ and $L_q^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ with $2 < q < \infty$. Section 2.3 deals with the interpolation of our Hardy spaces. In section 2.4, we show that our Hardy spaces coincide with those of [34]. So, we can give an explicit completely unconditional basis for the space $H_1(\mathbb{R})$, when $H_1(\mathbb{R})$ is equipped with an appropriate operator space structure.

We end this introduction by the convention that throughout the chapter the letter c will denote an absolute positive constant, which may vary from lines to lines, and c_p a positive constant depending only on p.

2.1 Preliminaries

2.1.1 Operator-valued noncommutative L_p-spaces

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ and $S_{\mathcal{M}}^+$ be the set of all positive element x in $\mathcal{M}$ with $\tau(s(x)) < \infty$, where $s(x)$ is the smallest projection e such that $exe = x$. Let $S_{\mathcal{M}}$ be the linear span of $S_{\mathcal{M}}^+$. Then any $x \in S_{\mathcal{M}}$ has finite trace, and $S_{\mathcal{M}}$ is a w^*-dense $*$-subalgebra of $\mathcal{M}$.

Let $1 \leq p < \infty$. For any $x \in S_{\mathcal{M}}$, the operator $|x|^p$ belongs to $S_{\mathcal{M}}^+$ ($|x| = (x^*x)^{\frac{1}{2}}$). We define

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}, \qquad \forall x \in S_{\mathcal{M}}.$$

One can check that $\|\cdot\|_p$ is well defined and is a norm on $S_{\mathcal{M}}$. The completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ is denoted by $L_p(\mathcal{M})$ which is the usual noncommutative L_p- space associated with $(\mathcal{M},\tau)$. For convenience, we usually set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$. The elements of $L_p(\mathcal{M},\tau)$ can be described as closed densely defined operators on H (H being the Hilbert space on which $\mathcal{M}$ acts). We refer the reader to [49] for more information on noncommutative L_p-spaces.

In this chapter, we are concerned with three operator-valued noncommutative L_p-spaces. The first one is the noncommutative space $L_p(\mathcal{M};\ell_2^c)$ (resp. $L_p(\mathcal{M};\ell_2^r)$), which is studied at length in [17]. For this space, we need the following properties. In the sequel, p' will always denote the conjugate index of p.

Lemma 1.1. *Let* $1 \leq p < \infty$. *Then*

$$(L_p(\mathcal{M};\ell_2^c))^* = L_{p'}(\mathcal{M};\ell_2^c). \tag{2.1.1}$$

Thus, for $f = (f_k)_k \in L_p(\mathcal{M};\ell_2^c)$ *and* $g = (g_k)_k \in L_{p'}(\mathcal{M};\ell_2^c)$, *we have*

$$|\tau(\langle f, g\rangle)| \leq \|f\|_{L_p(\mathcal{M};\ell_2^c)}\|g\|_{L_{p'}(\mathcal{M};\ell_2^c)},$$

where

$$\langle f, g\rangle = \sum_k f_k g_k^*.$$

Lemma 1.2. *Let* $1 \leq p_0 < p < p_1 \leq \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *Then*

$$[L_{p_0}(\mathcal{M};\ell_2^c), L_{p_1}(\mathcal{M};\ell_2^c)]_\theta = L_p(\mathcal{M};\ell_2^c). \tag{2.1.2}$$

A similar equality holds for row spaces.

The second one is the ℓ_∞-valued noncommutative space $L_p(\mathcal{M};\ell_\infty)$, which is studied by Pisier [47] for an injective $\mathcal{M}$ and Junge [16] for a general $\mathcal{M}$ (see also [25] and [27] for more properties). About this one, we need the following property:

Lemma 1.3. *Let* $1 \leq p < \infty$. *Then*

$$(L_p(\mathcal{M};\ell_1))^* = L_{p'}(\mathcal{M};\ell_\infty).$$

Thus, for $x = (x_n)_n \in L_p(\mathcal{M};\ell_1)$ *and* $y = (y_n)_n \in L_{p'}(\mathcal{M};\ell_\infty)$, *we have*

$$\Big|\sum_{n\geq 1}\tau(x_n y_n)\Big| \leq \|x\|_{L_p(\mathcal{M};\ell_1)}\|y\|_{L_{p'}(\mathcal{M};\ell_\infty)}, \tag{2.1.3}$$

where $L_p(\mathcal{M};\ell_1)$ *is the space of all sequences* $x = (x_n)_n$ *such that*

$$\|(x_k)_k\|_{L_p(\mathbb{N};\ell_1)} = \sup_{x_n = \sum_k a_{n,k}^* b_{n,k}} \|\sum_{n,\,k}|a_{n,k}|^2\|_p^{1/2}\|\sum_{n,\,k}|b_{n,k}|^2\|_p^{1/2}$$

The third one is $L_p(\mathcal{M};\ell_\infty^c)$ for $2 \leq p \leq \infty$, which was introduced in [7] and is related with the second one by

$$\|(x_n)_n\|_{L_p(\mathcal{M};\ell_\infty^c)} = \|(|x_n|^2)_n\|_{L_{\frac{p}{2}}(\mathcal{M};\ell_\infty)}.$$

And these are normed spaces by the following characterization

$$\|(x_n)_n\|_{L_p(\mathcal{M};\ell_\infty^c)} = \inf_{x_n = y_n a}\|(y_n)\|_{\ell_\infty(L_\infty(\mathcal{M}))}\|a\|_{L_p(\mathcal{M})}.$$

We need the interpolation results about these spaces (see [40]):

Lemma 1.4. *Let* $2 \leq p_0 < p < p_1 \leq \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *Then*

$$[L_{p_0}(\mathcal{M};\ell_\infty^c), L_{p_1}(\mathcal{M};\ell_\infty^c)]_\theta = L_p(\mathcal{M};\ell_\infty^c). \tag{2.1.4}$$

2.1.2 Operator-valued Hardy spaces

In this chapter, for simplicity, we denote $L_\infty(\mathbb{R})\bar{\otimes}\mathcal{M}$ by $\mathcal{N}$. As indicated in the introduction, one can observe that we have the following operator-valued Calderón identity

$$f(x) = \sum_{I\in\mathcal{D}} \langle f, w_I\rangle w_I(x), \tag{2.1.5}$$

which holds when $f \in L_2(\mathcal{N})$. As in the classical case, for $f \in S_{\mathcal{N}}$, we define the two Littlewood-Paley square functions as

$$S_c(f)(x) = \Big(\sum_{I\in\mathcal{D}} \frac{|\langle f, w_I\rangle|^2}{|I|}\mathbb{1}_I(x)\Big)^{\frac{1}{2}}. \tag{2.1.6}$$

$$S_r(f)(x) = \Big(\sum_{I\in\mathcal{D}} \frac{|\langle f^*, w_I\rangle|^2}{|I|}\mathbb{1}_I(x)\Big)^{\frac{1}{2}}. \tag{2.1.7}$$

For $1 \le p < \infty$, define

$$\|f\|_{\mathcal{H}_p^c} = \|S_c(f)\|_{L_p(\mathbb{N})},$$

$$\|f\|_{\mathcal{H}_p^r} = \|S_r(f)\|_{L_p(\mathbb{N})}.$$

These are norms, which can be seen easily from the space $L_p(\mathcal{N};\ell_2^c(\mathcal{D}))$. So we define the spaces $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathbb{R},\mathcal{M})$) as the completion of $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})})$ (resp. $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})})$. Now, we define the operator-valued Hardy spaces as follows: for $1 \le p < 2$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M}) = \mathcal{H}_p^c(\mathbb{R},\mathcal{M}) + \mathcal{H}_p^r(\mathbb{R},\mathcal{M}) \tag{2.1.8}$$

with the norm

$$\|f\|_{\mathcal{H}_p} = \inf\{\|g\|_{\mathcal{H}_p^c} + \|h\|_{\mathcal{H}_p^r} : f = g + h, g \in \mathcal{H}_p^c, h \in \mathcal{H}_p^r\}$$

and for $2 \le p < \infty$,

$$\mathcal{H}_p(\mathbb{R},\mathcal{M}) = \mathcal{H}_p^c(\mathbb{R},\mathcal{M}) \cap \mathcal{H}_p^r(\mathbb{R},\mathcal{M}) \tag{2.1.9}$$

with the norm defined as

$$\|f\|_{\mathcal{H}_p} = \max\{\|f\|_{\mathcal{H}_p^c}, \|f\|_{\mathcal{H}_p^r}\}.$$

We can identify $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ as a subspace of $L_p(\mathcal{N};\ell_2^c(\mathcal{D}))$, which is related with the two maps below.

Definition 1.5. (i) The embedding map Φ is defined from $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ to $L_p(\mathcal{N};\ell_2^c(\mathcal{D}))$ by

$$\Phi(f) = \Big(\frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}}\mathbb{1}_I\Big)_{I\in\mathcal{D}}. \tag{2.1.10}$$

(ii) The projection map Ψ is defined from $L_2(\mathcal{N};\ell_2^c(\mathcal{D}))$ to $\mathcal{H}_2^c(\mathbb{R},\mathcal{M})$ by

$$\Psi((g_I)) = \sum_{I\in\mathcal{D}} \int \frac{g_I}{|I|^{\frac{1}{2}}}\mathbb{1}_I dy \cdot w_I. \tag{2.1.11}$$

2.1.3 Operator-valued $\mathcal{BMO}$ spaces

For $\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, set

$$\|\varphi\|_{\mathcal{BMO}^c} = \sup_{J\in\mathcal{D}} \Big\| \big(\frac{1}{|J|} \sum_{I\subset J} |\langle \varphi, w_I\rangle|^2\big)^{\frac{1}{2}} \Big\|_{\mathcal{M}} \tag{2.1.12}$$

and

$$\|\varphi\|_{\mathcal{BMO}^r} = \|\varphi^*\|_{\mathcal{BMO}^c(\mathbb{R},\mathcal{M})}.$$

These are again norms modulo constant functions. Define

$$\mathcal{BMO}^c(\mathbb{R},\mathcal{M}) = \{\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^c} < \infty\}$$

and

$$\mathcal{BMO}^r(\mathbb{R},\mathcal{M}) = \{\varphi \in L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^r} < \infty\}$$

Now we define

$$\mathcal{BMO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \cap \mathcal{BMO}^r(\mathbb{R},\mathcal{M}).$$

As in the martingale case [25], we can also define $L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ for all $2 < p \leq \infty$. For $\varphi \in L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, set

$$\|\varphi\|_{L_p^c\mathcal{MO}} = \Big\| \big(\frac{1}{|I_k^x|} \sum_{I\subset I_k^x} |\langle \varphi, w_I\rangle|^2\big)_k \Big\|^{\frac{1}{2}}_{L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)} \tag{2.1.13}$$

and

$$\|\varphi\|_{L_p^r\mathcal{MO}} = \|\varphi^*\|_{L_p^c\mathcal{MO}},$$

where I_k^x denotes the unique dyadic interval with length 2^{-k+1} that containing x. We will use the convention adopted in [27] for the norm in $L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)$. Thus

$$\Big\| \big(\frac{1}{|I_k^x|} \sum_{I\subset I_k^x} |\langle \varphi, w_I\rangle|^2\big)_k \Big\|^{\frac{1}{2}}_{L_{\frac{p}{2}}(\mathcal{N};\ell_\infty)} = \Big\| {\sup_k}^+ \frac{1}{|I_k^x|} \sum_{I\subset I_k^x} |\langle \varphi, w_I\rangle|^2 \Big\|^{\frac{1}{2}}_{L_{\frac{p}{2}}(\mathcal{N})}$$

These are norms, which can be seen from the Banach spaces $L_p(\mathcal{N}\bar{\otimes}B(\ell_2(\mathcal{D})); \ell_\infty^c)$. Again, we can define

$$L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M}) = \{\varphi \in L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{L_p^c\mathcal{MO}} < \infty\}$$

and

$$L_p^r\mathcal{MO}(\mathbb{R},\mathcal{M}) = \{\varphi \in L_p(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{L_r^c\mathcal{MO}} < \infty\}$$

Define

$$L_p\mathcal{MO}(\mathbb{R},\mathcal{M}) = L_p^c\mathcal{MO}(\mathbb{R},\mathcal{M}) \cap L_p^r\mathcal{MO}(\mathbb{R},\mathcal{M}).$$

Note that $L_\infty^c\mathcal{MO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}^c(\mathbb{R},\mathcal{M})$. It is easy to check all the spaces we defined here with respect to the relevant norms are Banach spaces.

2.2 Duality

To prove the first two duality results in this section, we need the following noncommutative Doob's inequality from [16].

Let $(\mathcal{E}_n)_n$ be the conditional expectation with respect to a filtration $(\mathcal{N}_n)_n$ of $\mathcal{N}$.

Lemma 2.1. *Let $1 < p \leq \infty$ and $f \in L_p(\mathcal{N})$. Then*

$$\|{\sup_n}^+ \mathcal{E}_n(f)\|_{L_p(\mathcal{N})} \leq c_p \|f\|_{L_p(\mathcal{N})}. \tag{2.2.1}$$

Theorem 2.2. *We have*

$$(\mathcal{H}_1^c(\mathbb{R}, \mathcal{M}))^* = \mathcal{BMO}^c(\mathbb{R}, \mathcal{M}) \tag{2.2.2}$$

with equivalent norms. That is, every $\varphi \in \mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$ induces a continuous linear functional l_φ on $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$ by

$$l_\varphi(f) = \tau \int \varphi^* f, \quad \forall f \in S_{\mathcal{N}}. \tag{2.2.3}$$

Conversely, for every $l \in (\mathcal{H}_1^c(\mathbb{R}, \mathcal{M}))^*$, there exists a $\varphi \in \mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$ such that $l = l_\varphi$. Moreover,

$$c^{-1} \|\varphi\|_{\mathcal{BMO}^c} \leq \|l_\varphi\|_{(\mathcal{H}_1^c)^*} \leq c \|\varphi\|_{\mathcal{BMO}^c}$$

where $c > 0$ is a universal constant.

Similarly, the duality holds between $\mathcal{H}_1^r$ and $\mathcal{BMO}^r$, between $\mathcal{H}_1$ and $\mathcal{BMO}$ with equivalent norms.

In order to adapt the arguments in the martingale case, we need to define the truncated square functions for $n \in \mathbb{Z}$,

$$S_{c,n}(f)(x) = \Big(\sum_{k=-\infty}^{n} \sum_{|I|=2^{-k+1}} \frac{|\langle f, w_I \rangle|^2}{|I|} \mathbb{1}_I(x) \Big)^{\frac{1}{2}}.$$

Proof. Since $S_{\mathcal{N}}$ is dense in $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$, by an approximation argument, we only need to prove the inequality

$$|l_\varphi(f)| \leq c \|\varphi\|_{\mathcal{BMO}^c} \|f\|_{\mathcal{H}_1^c}$$

for $f \in S_{\mathcal{N}}$. By approximation we may assume that $S_{c,n}(f)(x)$ is invertible in $\mathcal{M}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then we have

$$\begin{aligned}
|l_\varphi(f)| &= |\tau \int \varphi^* f dx| \\
&= \Big| \sum_n \tau \int \sum_{|I|=2^{-n+1}} \langle \varphi, w_I \rangle^* w_I \sum_{|I'|=2^{-n+1}} \langle f, w_{I'} \rangle w_{I'} dx \Big| \\
&= \Big| \sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I \rangle^*}{|I|^{\frac{1}{2}}} \mathbb{1}_I \sum_{|I'|=2^{-n+1}} \frac{\langle f, w_{I'} \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_{I'} dx \Big| \\
&\leq \sum_n \Big(\tau \int \Big| \sum_{|I|=2^{-n+1}} \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \Big|^2 S_{c,n}^{-1}(f) \Big)^{\frac{1}{2}} \\
&\qquad \cdot \Big(\tau \int \Big| \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \Big|^2 S_{c,n}(f) \Big)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big(\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{|\langle f, w_I\rangle|^2}{|I|} \mathbb{1}_I S_{c,n}^{-1}(f)\Big)^{\frac{1}{2}} \\
&\quad \cdot \Big(\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I\rangle|^2}{|I|} \mathbb{1}_I S_{c,n}(f)\Big)^{\frac{1}{2}} \\
&= A \cdot B.
\end{aligned}
$$

In the above estimates, the first equality has used the orthogonality of the w_I's on different levels, the second one the orthogonality of the w_I's on the same level and the disjoint of different dyadic I's on the same level; the first inequality has used the Hölder inequality in Lemma 1.1, and the second one the Cauchy-Schwarz inequality and the disjointness of different I's on the same level.

Now, let us estimate A:

$$
\begin{aligned}
A^2 &= \sum_n \tau \int (S_{c,n}^2(f) - S_{c,n-1}^2(f)) S_{c,n}^{-1}(f) \\
&= \sum_n \tau \int (S_{c,n}(f) - S_{c,n-1}(f))(1 + S_{c,n-1}(f) S_{c,n}^{-1}(f)) \\
&\leq \sum_n \tau \int (S_{c,n}(f) - S_{c,n-1}(f)) \|1 + S_{c,n-1}(f) S_{c,n}^{-1}(f)\|_\infty \\
&\leq 2 \sum_n \tau \int (S_{c,n}(f) - S_{c,n-1}(f)) \\
&= 2\|f\|_{\mathcal{H}_1^c}.
\end{aligned}
$$

For the first inequality, we have used the Hölder inequality and the positivity of $S_{c,n}(f) - S_{c,n-1}(f)$.

The second term is estimated as follows:

$$
\begin{aligned}
B^2 &= \sum_k \tau \int (S_{c,k}(f) - S_{c,k-1}(f)) \sum_{n \geq k} \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I\rangle|^2}{|I|} \mathbb{1}_I \\
&= \sum_k \tau \sum_j (S_{c,k}(f) - S_{c,k-1}(f)) \int_{I_k^j} \sum_{n \geq k} \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I\rangle|^2}{|I|} \mathbb{1}_I \\
&= \sum_k \tau \sum_j \int_{I_k^j} (S_{c,k}(f) - S_{c,k-1}(f)) \frac{1}{|I_k^j|} \sum_{I \subset I_k^j} |\langle \varphi, w_I\rangle|^2 \\
&\leq \sum_k \sum_j \tau \int_{I_k^j} (S_{c,k}(f) - S_{c,k-1}(f)) \Big\| \frac{1}{|I_k^j|} \sum_{I \subset I_k^j} |\langle \varphi, w_I\rangle|^2 \Big\|_\infty \\
&\leq \|\varphi\|^2_{\mathcal{BMO}^c} \sum_k \sum_j \tau \int_{I_k^j} (S_{c,k}(f) - S_{c,k-1}(f)) \\
&= \|\varphi\|^2_{\mathcal{BMO}^c} \|f\|_{\mathcal{H}_1^c}
\end{aligned}
$$

The fist equality has used the Fubini theorem, the second one the fact that $S_{c,k-1}(f)$ and $S_{c,k}(f)$ are constant on the dyadic interval $I_k^j = [j2^{-k+1}, (j+1)2^{-k+1})$; the first inequality has used the Hölder inequality and the positivity of $S_{c,n}(f) - S_{c,n-1}(f)$.

Now, let us begin to deal with the other direction, i.e. suppose that l is a bounded linear functional on $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$, we want to find an operator-valued function φ in $\mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$, such that $l = l_\varphi$ and $l_\varphi(f) = \tau\int \varphi^* f$ for $f \in S_{\mathcal{N}}$. By the embedding operator Φ in (2.1.10) and by the Hahn-Banach theorem, l extends to a bounded continuous functional on $L_1(\mathcal{N};\ell_2^c(\mathcal{D}))$ of the same norm. Then by the results in Lemma 1.1 there exists $g = (g_I)_{I\in\mathcal{D}}$ such that $\|g\|_{L_\infty(\mathcal{N};\ell_2^c(\mathcal{D}))} = \|l\|$, and

$$l(f) = \tau \int \sum_{I\in\mathcal{D}} g_I^* \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I, \quad \forall f \in S_{\mathcal{N}}.$$

Now let $\varphi = \Psi(g)$, where Ψ is defined as (2.1.11). The orthogonality of the w_I's yields

$$\begin{aligned}\|\sum_{I\subset J} |\langle \varphi, w_I\rangle|^2\|_{\mathcal{M}} &= \|\sum_{I\subset J} |\int \frac{g_I}{|I|^{\frac{1}{2}}}\mathbb{1}_I|^2\|_{\mathcal{M}} \leq \|\sum_{I\subset J}\int_J |g_I|^2\|_{\mathcal{M}} \\ &\leq |J| \|\sum_{I\subset J} |g_I|^2\|_{L_\infty(\mathcal{N})} \leq |J| \|(g_I)_I\|_{L_\infty(\mathcal{N};\ell_2^c(\mathcal{D}))},\end{aligned}$$

where the first inequality has used the Kadison-Schwartz inequality. Also thanks to the orthogonality of the w_I's, we get

$$l(f) = \tau \int \sum_{I\in\mathcal{D}} g_I^* \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I = \tau\int \varphi^* f$$

for all $f \in S_{\mathcal{N}}$. Therefore, we complete the proof about $\mathcal{H}_1^c(\mathbb{R},\mathcal{M})$ and $\mathcal{BMO}^c(\mathbb{R},\mathcal{M})$. Passing to adjoint, we have the conclusion concerning $\mathcal{H}_1^r$ and $\mathcal{BMO}^r$. Finally, by the classical fact that the dual of a sum space is the intersection space, we obtain the duality between $\mathcal{H}_1$ and $\mathcal{BMO}$. □

Theorem 2.3. *Let $1 < p < 2$. We have*

$$(\mathcal{H}_p^c(\mathbb{R},\mathcal{M}))^* = L_{p'}^c\mathcal{MO}(\mathbb{R},\mathcal{M}) \tag{2.2.4}$$

with equivalent norms. That is, every $\varphi \in L_{p'}^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ induces a continuous linear functional l_φ on $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ by

$$l_\varphi(f) = \tau\int \varphi^* f, \quad \forall f \in S_{\mathcal{N}}. \tag{2.2.5}$$

Conversely, for every $l \in (\mathcal{H}_p^c(\mathbb{R},\mathcal{M}))^*$, there exists an operator-valued function $\varphi \in L_{p'}^c\mathcal{MO}(\mathbb{R},\mathcal{M})$ such that $l = l_\varphi$ and

$$c_p^{-1}\|\varphi\|_{L_{p'}^c\mathcal{MO}} \leq \|l_\varphi\|_{(\mathcal{H}_p^c)^*} \leq \sqrt{2}\|\varphi\|_{L_{p'}^c\mathcal{MO}}$$

Similarly, the duality holds between $\mathcal{H}_p^r$ and $L_{p'}^r$, between $\mathcal{H}_p$ and $L_{p'}\mathcal{MO}$ with equivalent norms.

We need the following lemma of [25]. We write it down for the reader's convenience but without proof.

Lemma 2.4. *Let s,t be two real numbers such that $s < t$ and $0 \leq s \leq 1 \leq t \leq 2$. Let x,y be two positive operators such that $x \leq y$ and $x^{t-s}, y^{t-s} \in L_1(\mathbb{N})$. Then*

$$\tau\int y^{-s/2}(y^t - x^t)y^{-s/2} \leq 2\tau\int y^{-(s+1-t)/2}(y-x)y^{-(s+1-t)/2}.$$

Proof. We need only to prove the first assertion on $\mathcal{H}_p^c$. Since $S_{\mathcal{N}}$ is dense in $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$, by an approximation argument, we only need to prove the inequality

$$|l_\varphi(f)| \leq c\|\varphi\|_{L^c_{p'}\mathcal{MO}}\|f\|_{\mathcal{H}_p^c}$$

for $f \in S_{\mathcal{N}}$. By approximation we may assume that $S_{c,n}(f)(x)$ is invertible in $\mathcal{M}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. By the similar principle as in the noncommutative martingale case as in [25], we have

$$\begin{aligned}
|l_\varphi(f)| &= |\tau \int \varphi^* f dx| \\
&= \Big|\sum_n \tau \int \sum_{|I|=2^{-n+1}} \langle \varphi, w_I\rangle^* w_I \sum_{|I'|=2^{-n+1}} \langle f, w_{I'}\rangle w_{I'} dx\Big| \\
&= \Big|\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I\rangle^*}{|I|^{\frac12}} \mathbb{1}_I \sum_{|I'|=2^{-n+1}} \frac{\langle f, w_{I'}\rangle}{|I|^{\frac12}} \mathbb{1}_{I'} dx\Big| \\
&\leq \sum_n \Big(\tau \int \Big| \sum_{|I|=2^{-n+1}} \frac{\langle f, w_I\rangle}{|I|^{\frac12}} \mathbb{1}_I\Big|^2 S_{c,n}^{p-2}(f)\Big)^{\frac12} \\
&\qquad \cdot \Big(\tau \int \Big| \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I\rangle}{|I|^{\frac12}} \mathbb{1}_I\Big|^2 S_{c,n}^{2-p}(f)\Big)^{\frac12} \\
&\leq \Big(\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{|\langle f, w_I\rangle|^2}{|I|} \mathbb{1}_I S_{c,n}^{p-2}(f)\Big)^{\frac12} \\
&\qquad \cdot \Big(\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I\rangle|^2}{|I|} \mathbb{1}_I S_{c,n}^{2-p}(f)\Big)^{\frac12} \\
&= A \cdot B.
\end{aligned}$$

Now we need the above lemma to estimate the first term. Take $s = 2-p$ and $t = 2$, the lemma yields

$$\begin{aligned}
A^2 &= \sum_n \tau \int (S_{c,n}^2(f) - S_{c,n-1}^2(f)) S_{c,n}^{p-2}(f) \\
&= \sum_n \tau \int S_{c,n}^{-(2-p)/2}(f)(S_{c,n}^2(f) - S_{c,n-1}^2(f)) S_{c,n}^{-(2-p)/2}(f) \\
&\leq 2\sum_n \tau \int S_{c,n}^{-(1-p)/2}(f)(S_{c,n}(f) - S_{c,n-1}(f)) S_{c,n}^{-(1-p)/2}(f) \\
&= 2\sum_n \tau \int S_{c,n}(f) - S_{c,n-1}(f) S_{c,n}^{p-1}(f) \\
&\leq 2\sum_n \tau \int S_{c,n}^p(f) - S_{c,n-1}^p(f) \\
&= 2\|f\|_{\mathcal{H}_p^c}^p.
\end{aligned}$$

The last inequality has used two elementary inequalities: $0 \leq S_{c,n-1}(f) \leq S_{c,n}(f)$ implies $S_{c,n-1}^{p-1}(f) \leq S_{c,n}^{p-1}(f)$ for $0 < p-1 < 1$; and $\tau(S_{c,n-1}^p(f)) \leq \tau(S_{c,n-1}^{\frac12}(f) S_{c,n}^{p-1}(f) S_{c,n-1}^{\frac12}(f))$.

The second term can be deduced from the nontrivial duality results in Lemma 2.1.3 for $1 < p < \infty$ as follows.

$$
\begin{aligned}
B^2 &= \sum_k \tau \int S_{c,k}^{2-p}(f) - S_{c,k-1}^{2-p}(f) \sum_{n\geq k} \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I\rangle|^2}{|I|} \mathbb{1}_I \\
&= \sum_k \tau \sum_j S_{c,k}^{2-p}(f) - S_{c,k-1}^{2-p}(f) \int_{I_k^j} \sum_{n\geq k} \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I\rangle|^2}{|I|} \mathbb{1}_I \\
&= \sum_k \tau \sum_j \int \mathbb{1}_{I_k^j}(x) S_{c,k}^{2-p}(f)(x) - S_{c,k-1}^{2-p}(f)(x) \frac{1}{|I_k^j|} \sum_{I\subset I_k^j} |\langle \varphi, w_I\rangle|^2 dx \\
&= \sum_k \tau \int S_{c,k}^{2-p}(f)(x) - S_{c,k-1}^{2-p}(f)(x) \frac{1}{|I_k^x|} \sum_{I\subset I_k^x} |\langle \varphi, w_I\rangle|^2 dx \\
&\leq \| \sum_k S_{c,k}^{2-p}(f) - S_{c,k-1}^{2-p}(f) \|_{L_{(p'/2)'}} \Big\| \sup_k \frac{1}{|I_k^x|} \sum_{I\subset I_k^x} |\langle \varphi, w_I\rangle|^2 \Big\|_{L_{p'/2}} \\
&= \|\varphi\|^2_{L^c_{p'}\mathcal{MO}} \|f\|^{2-p}_{\mathcal{H}_p^c}
\end{aligned}
$$

The fist equality has used the Fubini theorem, the second one the fact that $S_{c,k-1}(f)$ and $S_{c,k}(f)$ are constant on the dyadic intervals with length 2^{-k+1}.

For the other direction, we can carry out the proof as that in the case $p = 1$. Suppose that l is a bounded linear functional on $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$. By the embedding operator Φ and by Hahn-Banach theorem, and the results in Lemma 1.1, we can find $g = (g_I)_{I\in\mathcal{D}}$ such that $\|g\|_{L_{p'}(\mathcal{N};\ell_2^c(\mathcal{D}))} = \|l\|$ and

$$
l(f) = \tau \int \sum_{I\in\mathcal{D}} g_I^* \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I, \forall f \in S_{\mathcal{N}}.
$$

Now let $\varphi = \Psi(g)$ defined in (2.1.11), the orthogonality of the w_I's yields

$$
\begin{aligned}
&\Big\|\sup_n{}^+ \frac{1}{|I_n^x|} \sum_{I\subset I_n^x} |\langle \varphi, w_I\rangle|^2 \Big\|_{L_{p'/2}(\mathcal{N})} \\
&\quad = \Big\|\sup_n{}^+ \frac{1}{|I_n^x|} \sum_{I\subset I_n^x} \Big| \int \frac{g_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I \Big|^2 \Big\|_{L_{p'/2}(\mathcal{N})} \\
&\quad \leq \Big\|\sup_n{}^+ \frac{1}{|I_n^x|} \sum_{I\subset I_n^x} \int_{I_n^x} |g_I|^2 \Big\|_{L_{p'/2}(\mathcal{N})} \\
&\quad \leq \Big\|\sup_n{}^+ \frac{1}{|I_n^x|} \int_{I_n^x} \sum_{I\subset I_n^x} |g_I|^2 \Big\|_{L_{p'/2}(\mathcal{N})} \\
&\quad \leq \Big\|\sup_n{}^+ \frac{1}{|I_n^x|} \int_{I_n^x} \sum_{I\in\mathcal{D}} |g_I|^2 \Big\|_{L_{p'/2}(\mathcal{N})} \\
&\quad \leq c \Big\| \sum_{I\in\mathcal{D}} |g_I|^2 \Big\|_{L_{p'/2}(\mathcal{N})} \\
&\quad = c \Big\|(g_I)_I\Big\|_{L_{p'}(\mathcal{N};\ell_2^c(\mathcal{D}))},
\end{aligned}
$$

where for the first inequality we have used the Kadison-Schwartz inequality, and the last inequality is (2.2.1). Also due to the orthogonality of the w_I's, we get

$$
l(f) = \tau \int \sum_{I\in\mathcal{D}} g_I^* \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I = \tau \int \varphi^* f,
$$

for all $f \in S_{\mathcal{N}}$. Therefore, we complete the proof about $\mathcal{H}_p^c(\mathbb{R},\mathcal{M})$ and $L_{p'}^c\mathcal{MO}(\mathbb{R},\mathcal{M})$. $\square$

Instead of using the noncommutative Doob's inequality, we will use the following noncommutative Stein inequality from [48] to prove the duality between the spaces $\mathcal{H}_p^c$, $1 < p < \infty$.

Let $(\mathcal{E}_n)_n$ be the conditional expectation with respect to a filtration $(\mathcal{N}_n)_n$ of $\mathcal{N}$.

Lemma 2.5. *Let $1 < p < \infty$ and $a = (a_n)_n \in L_p(\mathcal{N};\ell_2^c)$. Then there exists a constant depending only on p such that*

$$\Big\|\big(\sum_n |\mathcal{E}_n a_n|^2\big)^{\frac12}\Big\|_{L_p(\mathcal{N})} \le c_p\Big\|\big(\sum_n |a_n|^2\big)^{\frac12}\Big\|_{L_p(\mathcal{N})}. \tag{2.2.6}$$

Theorem 2.6. *For any $1 < p < \infty$, we have*

$$(\mathcal{H}_p^c(\mathbb{R},\mathcal{M}))^* = \mathcal{H}_{p'}^c(\mathbb{R},\mathcal{M}), \tag{2.2.7}$$

Proof. By a similar reasoning as in the corresponding part of the proof of Theorem 2.2, we can carry out the following calculation,

$$\begin{aligned}
|l_\varphi(f)| &= |\tau\int \varphi^* f dx| \\
&= \Big|\sum_n \tau\int \sum_{|I|=2^{-n+1}} \langle\varphi, w_I\rangle^* w_I \sum_{|I'|=2^{-n+1}} \langle f, w_{I'}\rangle w_{I'} dx\Big| \\
&= \Big|\sum_n \tau\int \sum_{|I|=2^{-n+1}} \frac{\langle\varphi, w_I\rangle^*}{|I|^{\frac12}}\mathbb{1}_I \frac{\langle f, w_I\rangle}{|I|^{\frac12}}\mathbb{1}_I dx\Big| \\
&\le \|\big(\sum_{I\in\mathcal{D}} \frac{|\langle f, w_I\rangle|^2}{|I|}\mathbb{1}_I\big)^{\frac12}\|_{L_p(\mathbb{R},\mathcal{M})} \cdot \|\big(\sum_{I\in\mathcal{D}} \frac{|\langle\varphi, w_I\rangle|^2}{|I|}\mathbb{1}_I\big)^{\frac12}\|_{L_{p'}(\mathbb{R},\mathcal{M})}.
\end{aligned}$$

Now, we turn to the proof of the inverse direction. Take a bounded linear functional $l \in (\mathcal{H}_p^c(\mathbb{R},\mathcal{M}))^*$, by the embedding operator Φ and the Hahn-Banach extension theorem, l extends to a bounded linear functional on $L_p(\mathcal{N};\ell_2^c)$ with the same norm. Thus by (1.1), there exists a sequence $g = (g_I)_I$ such that

$$\|g\|_{L_q(\mathcal{N};l_2^c(\mathcal{D}))} = \|l\|$$

and

$$l(f) = \tau\int \sum_{I\in\mathcal{D}} g_p^* \frac{\langle f, w_I\rangle}{|I|^{\frac12}}\mathbb{1}_I, \forall f \in S_{\mathcal{N}}.$$

Now let $\varphi = \Psi(g)$ where Ψ is defined in (2.1.11), then applying the Stein inequality (2.5) to the conditional expectation

$$\mathcal{E}_I(h) = \sum_J \frac{1}{|J|}\int_J h(y)dy \cdot \mathbb{1}_J,$$

where J is dyadic interval with the same length as I, we get

$$\begin{aligned}
\|\varphi\|_{\mathcal{H}_{p'}^c(\mathbb{R},\mathcal{M})} &= \|\big(\sum_{I\in\mathcal{D}} |\frac{1}{|I|}\int_I g_I dy \cdot \mathbb{1}_I|^2\big)^{\frac12}\|_{L_{p'}(\mathcal{N})} \\
&\le \|\big(\sum_{I\in\mathcal{D}} |\mathcal{E}_I(g_I)|^2\big)^{\frac12}\|_{L_{p'}(\mathcal{N})} \\
&\le c_{p'}\|\big(\sum_{I\in\mathcal{D}} |g_I|^2\big)^{\frac12}\|_{L_{p'}(\mathcal{N})}.
\end{aligned}$$

By the orthogonality of the w_I's, we have

$$l(f) = \tau \int \sum_{I \in \mathcal{D}} g_I^* \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I = \tau \int \varphi^* f,$$

for all $f \in S_{\mathcal{N}}$. □

From the proof of the second part of Theorem 2.2, Theorem 2.3 and Theorem 2.6, we state the boundedness of Ψ as a corollary.

Corollary 2.7. (i) *Let $1 < p < \infty$, Ψ is a projection map from $L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$ onto $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ if we identify the latter as a subspace of the former.*

(ii) *Let $2 < p \leq \infty$, Ψ is also a bounded map from $L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$ to $L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M})$.*

Theorem 2.3 and Theorem 2.6 immediately imply the following corollary:

Corollary 2.8. *Let $2 < p < \infty$. Then*

$$\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}) = L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M}), \quad \forall 2 < p < \infty$$

with equivalent norms.

However, for the part $L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M}) \subset \mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$, we can give another proof. The idea is essentially similar to that in [34], the good news is that in our case, the argument seems very elegant. Now we give the detailed proof.

proof. Our tent space is defined as

$$T_p^c = \Big\{ f = \{f_I\}_I \in L_p(\mathcal{M}; \ell_2^c(\mathcal{D})) : \quad \tau \int \big(\sum_{I \in \mathcal{D}} \frac{f_I^2}{|I|} \mathbb{1}_I \big)^{\frac{p}{2}} < \infty \Big\}$$

We claim that every $\varphi \in L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M})$ induces a bounded linear functional on $T_{p'}^c$,

$$l_\varphi(f) = \tau \int \sum_{I \in \mathcal{D}} \frac{\langle \varphi, w_I \rangle^*}{|I|^{\frac{1}{2}}} \mathbb{1}_I \frac{f_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I dx$$

and $\|l_\varphi\| \leq \|\varphi\|_{L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M})}$. The proof is just the copy of the proof of the first part in the last theorem. Now $T_{p'}^c$ is naturally embedded into $L_{p'}(\mathcal{N}; \ell_2^c(\mathcal{D}))$ by $(f_I)_I \to (\frac{f_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I)_I$. So by the Hahn-Banach extension theorem, l_φ extends to an bounded linear functional on $L_{p'}(\mathcal{N}; \ell_2^c(\mathcal{D}))$ with the same norm. Then by the duality between

$$(L_{p'}(\mathcal{N}; \ell_2^c(\mathcal{D})))^* = L_p(\mathcal{N}; \ell_2^c(\mathcal{D})).$$

there exists a unique $h = (h_I)_I$ such that $\|h\|_{L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))} \leq \|l_\varphi\|$ and for $f = (f_I)_I \in T_{p'}^c$,

$$l_\varphi(f) = \tau \int \sum_{I \in \mathcal{D}} h_I^* \frac{f_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I dx$$

So we get

$$\frac{\langle \varphi, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I = h_I,$$

thus,

$$\begin{aligned} \|\varphi\|_{\mathcal{H}_p^c} &= \Big\| \big(\sum_{I \in \mathcal{D}} \frac{\langle \varphi, w_I \rangle^*}{|I|^{\frac{1}{2}}} \mathbb{1}_I \big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{N})} \\ &= \|h_I\|_{L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))} \leq \|l_\varphi\| \end{aligned}$$

□

2.3 Interpolation

This section is devoted to the interpolation of our wavelet Hardy spaces. The interpolation results below will be needed in the next section to compare our Hardy spaces with those of Mei.

Lemma 3.1. *Let $1 < p_0 < p < p_1 < \infty$, we have*

$$[\mathcal{H}^c_{p_0}(\mathbb{R}, \mathcal{M}), \mathcal{H}^c_{p_1}(\mathbb{R}, \mathcal{M})]_\theta = \mathcal{H}^c_p(\mathbb{R}, \mathcal{M}) \tag{2.3.1}$$

with equivalent norms, where θ satisfies $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. The embedding map Φ yields

$$[\mathcal{H}^c_{p_0}, \mathcal{H}^c_{p_1}]_\theta \subset \mathcal{H}^c_p.$$

On the other hand, it is the boundedness of the projection map Ψ from $L_p(\mathbb{N}; \ell^c_2(\mathcal{D}))$ to $\mathcal{H}^c_p(\mathbb{R}, \mathcal{M})$ stated in Corollary 2.7 that yields the inverse direction. ☐

Theorem 3.2. *Let $1 \leq q < p < \infty$, we have*

$$[\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), \mathcal{H}^c_q(\mathbb{R}, \mathcal{M})]_{\frac{q}{p}} = \mathcal{H}^c_p(\mathbb{R}, \mathcal{M}) \tag{2.3.2}$$

with equivalent norms.

Proof. We will prove the theorem by a general strategy as appeared in [40].

Step 1: We prove the conclusion for $2 < q < p < \infty$:

$$[\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), \mathcal{H}^c_q(\mathbb{R}, \mathcal{M})]_{\frac{q}{p}} = \mathcal{H}^c_p(\mathbb{R}, \mathcal{M}). \tag{2.3.3}$$

The identity can be seen easily from the following two inclusions. On one hand, the operator Φ which in (2.1.10), together with (2.1.2) yields

$$[\mathcal{H}^c_1(\mathbb{R}, \mathcal{M}), \mathcal{H}^c_{q'}(\mathbb{R}, \mathcal{M})]_{\frac{q}{p}} \subset \mathcal{H}^c_{p'}(\mathbb{R}, \mathcal{M}).$$

Then by duality and Corollary 2.8, we have

$$L^c_p\mathcal{MO}(\mathbb{R}, \mathcal{M}) \subset [\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), L^c_q\mathcal{MO}(\mathbb{R}, \mathcal{M})]_{\frac{q}{p}}. \tag{2.3.4}$$

On the other hand, the operator $\mathcal{T}$ identifies the space $L^c_p\mathcal{MO}(\mathbb{R}, \mathcal{M})$ as a subspace of $L_p(L_\infty(\mathcal{N}\bar{\otimes}B(\ell_2(\mathcal{D})); \ell^c_\infty)$ defined by

$$\mathcal{T}(\varphi) = \langle f, w_I \rangle |I^t_k|^{-\frac{1}{2}} \mathbb{1}_{I \subset I^t_k}(I) \otimes e_{I,1}, \tag{2.3.5}$$

together with Lemma 1.4 yields

$$[\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), L^c_q\mathcal{MO}(\mathbb{R}, \mathcal{M})]_{\frac{q}{p}} \subset L^c_p\mathcal{MO}(\mathbb{R}, \mathcal{M}). \tag{2.3.6}$$

Step 2: we prove the conclusion for $1 < q < p < \infty$. This step can be divided into two substeps.

Substep 21: $p > 2$. Let $p < s < \infty$. By Step 1, we have

$$[\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), \mathcal{H}^c_p(\mathbb{R}, \mathcal{M})]_{\frac{p}{s}} = \mathcal{H}^c_s(\mathbb{R}, \mathcal{M}).$$

On the other hand, by Theorem 3.1, we have

$$[\mathcal{H}_q^c, \mathcal{H}_s^c]_\theta = \mathcal{H}_p^c,$$

where(and in the rest of the paper) θ denote the interpolation parameter. Then Wolff's interpolation theorem yields the result.

Substep 22: $p \leq 2$. Let $s > 2$, then by Substep 21, we have

$$[\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), \mathcal{H}_p^c(\mathbb{R}, \mathcal{M})]_{\frac{p}{s}} = \mathcal{H}_s^c(\mathbb{R}, \mathcal{M}).$$

Then together with Lemma 3.1, Wolff's interpolation theorem yields the result.

Step 3: we prove the conclusion for $1 = q < p < \infty$. Take $s > \max(p, 2)$. By Step 2 and duality [3, Theorem 4.3.1], we get

$$[\mathcal{H}_1^c, \mathcal{H}_s^c]_\theta = \mathcal{H}_p^c.$$

Then together with Step 2, Wolff's interpolation yields the conclusion. □

Remark 3.3. If one can directly prove Lemma 3.1 for $p_0 = 1$, we can prove the above theorem without the help of $L_p^c\mathcal{MO}(\mathbb{R}, \mathcal{M})$ for $2 < p < \infty$ as carried out in [2], where one needs an auxiliary space.

Theorem 3.4. *For $1 < p < \infty$, we have*

$$\mathcal{H}_p(\mathbb{R}, \mathcal{M}) = L_p(\mathcal{N})$$

with equivalent norms.

Proof. There are several ways to prove this result. One can prove it by the strategy in [48] together with Stein's inequality (2.5). Here, we just use the fact that $L_p(\mathcal{M})$ with $1 < p < \infty$ is a UMD space and our $(w_I)_I$ is an complete orthonormal basis. So by Theorem 3.8 in [15], we have

$$\|f\|_{L_p(\mathcal{N})} \simeq \Big(\mathbb{E}\Big\| \sum_{I\in\mathcal{D}} \varepsilon_I \frac{\langle f, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \Big\|_{L_p(\mathcal{N})}^p\Big)^{\frac{1}{p}}.$$

Then we complete the proof for $2 \leq p < \infty$ by Khintchine's inequalities. Now, let us prove the case $1 < p < 2$. Let $f \in \mathcal{H}_p(\mathbb{R}, \mathcal{M})$, then for any $\epsilon > 0$, by the definition of $\mathcal{H}_p(\mathbb{R}, \mathcal{M})$, there exists a decomposition $f = f_c + f_r$ such that

$$\|f_c\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})} + \|f_r\|_{\mathcal{H}_p^r(\mathbb{R},\mathcal{M})} \leq \|f\|_{\mathcal{H}_p(\mathbb{R},\mathcal{M})} + \epsilon.$$

Take any $g \in L_{p'}(\mathcal{N})$, by the results for $p' > 2$, the operator-valued Calderón identity (2.1.5) yields

$$\begin{aligned}
|\tau \int g f^*| &= |\sum_{I\in\mathcal{D}} \tau \int \frac{\langle g, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \cdot \frac{\langle f^*, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I| \\
&\leq |\sum_{I\in\mathcal{D}} \tau \int \frac{\langle g, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \cdot \frac{\langle f_c^*, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I| \\
&\quad + |\sum_{I\in\mathcal{D}} \tau \int \frac{\langle g, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \cdot \frac{\langle f_r^*, w_I\rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I| \\
&\leq \|S_c(g)\|_{L_{p'}(\mathcal{N})} \|S_c(f_c)\|_{L_p(\mathcal{N})} + |S_r(g)\|_{L_{p'}(\mathcal{N})} \|S_r(f_r)\|_{L_p(\mathcal{N})}
\end{aligned}$$

$$\leq c_{p'} \|g\|_{L_{p'}} (\|f\|_{\mathcal{H}_p(\mathbb{R},\mathcal{M})} + \epsilon).$$

Taking sup and let $\epsilon \to 0$, we get the required result.

Finally, we prove the inverse inequality. Let $f \in L_p(\mathcal{N})$, by duality, we can find two sequences of functions $(F_{c,I})_I \in L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$ and $(F_{r,I})_I \in L_p(\mathcal{N}; \ell_2^r(\mathcal{D}))$ such that $F_{c,I} + F_{r,I} = \langle f, w_I \rangle |I|^{-\frac{1}{2}} \mathbb{1}_I$ and

$$\|(F_{c,I})_I\|_{L_p(\mathcal{N};\ell_2^c(\mathcal{D}))} + \|(F_{r,I})_I\|_{L_p(\mathcal{N};\ell_2^r(\mathcal{D}))} \leq \|f\|_{L_p(\mathcal{N})}.$$

Let $f_c = \Psi((F_{c,I})_I)$ and $f_r = \Psi((F_{r,I})_I)$, by identity (2.1.5), we have $f = f_c + f_r$. On the other hand, by the Stein inequality (2.5), we have $\|f_c\|_{\mathcal{H}_p^c(\mathbb{R},\mathcal{M})} \leq \|(F_{c,I})_I\|_{L_p(\mathcal{N};\ell_2^c(\mathcal{D}))}$ and $\|f_r\|_{\mathcal{H}_p^r(\mathbb{R},\mathcal{M})} \leq \|(F_{r,I})_I\|_{L_p(\mathcal{N};\ell_2^r(\mathcal{D}))}$. So we have found the desired decomposition of f. $\square$

Theorem 3.5. *The following results hold with equivalent norms:*

(i) *Let $1 \leq q < p < \infty$, we have*

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}} = L_p(\mathcal{N}). \tag{2.3.7}$$

(ii) *Let $1 < q < p \leq \infty$, we have*

$$[\mathcal{H}_1(\mathbb{R},\mathcal{M}), L_p(\mathcal{N})]_{\frac{p'}{q'}} = L_q(\mathcal{N}). \tag{2.3.8}$$

(iii) *Let $1 < p < \infty$, we have*

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), \mathcal{H}_1(\mathbb{R},\mathcal{M})]_{\frac{1}{p}} = L_p(\mathcal{N}). \tag{2.3.9}$$

In order to prove this theorem, we need the following result from the theory of interpolation. We formulate it here without proof.

Lemma 3.6. *Let A_0, B_0, A_1, B_1 be four Banach spaces satisfying the property needed for interpolation. Then*

$$[A_0 + B_0, A_1 + B_1]_\theta \supset [A_0, A_1]_\theta + [B_0, B_1]_\theta$$

and

$$[A_0 \cap B_0, A_1 \cap B_1]_\theta \subset [A_0, A_1]_\theta \cap [B_0, B_1]_\theta.$$

Proof. (i) We also exploit the similar but different strategy with that in the proof of Theorem 3.2.

Step 1: we prove the results for $2 \leq q < p < \infty$. By Theorem 3.4, Theorem 3.2 and the lemma, we have

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}} \subset L_p(\mathcal{N}).$$

The inverse direction follows from $L_\infty(\mathcal{N}) \subset \mathcal{BMO}(\mathbb{R},\mathcal{M})$,

$$\begin{aligned} L_p(\mathcal{N}) &= [L_\infty(\mathcal{N}), L_q(\mathcal{N})]_{\frac{q}{p}} \\ &\subset [\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}} \end{aligned}$$

Step 2: we prove the results for $1 \leq q < 2 \leq p < \infty$. By Step 1, we have

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_2(\mathcal{N})]_{\frac{2}{p}} = L_p(\mathcal{N}).$$

Together with

$$L_2(\mathcal{N}) = [L_p(\mathcal{N}), L_q(\mathcal{N})]_\theta,$$

Wolff's interpolation yields the conclusion.

Step 3: we prove the results for $1 \le q < p < 2$. By Step 2, we have

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_p(\mathcal{N})]_{\frac{p}{2}} = L_2(\mathcal{N}).$$

Together with

$$L_p(\mathcal{N}) = [L_2(\mathcal{N}), L_q(\mathcal{N})]_\theta,$$

Wolff's interpolation yields the conclusion.

(ii) The results for $1 < q < p < \infty$ can be immediately proved by duality and the partial results in (i). For $p = \infty$, take $q < s < \infty$, then by Wolff's argument, we get the conclusion.

(iii) First, we prove conclusion for $p < 2$. Then by (i) and (ii), we have

$$[\mathcal{BMO}(\mathbb{R},\mathcal{M}), L_p(\mathcal{N})]_{\frac{p}{p'}} = L_{p'}(\mathcal{N})$$

and

$$[\mathcal{H}_1(\mathbb{R},\mathcal{M}), L_{p'}(\mathcal{N})]_{\frac{p}{p'}} = L_p(\mathcal{N}).$$

Therefore, we end with Wolff's argument. Second, the proof for $p > 2$ is the same. At last, when $p = 2$, we can take $s > 2$, by the results for $p \neq 2$ and reiteration theorem in [3, Theorem 4.6.1], we get

$$\begin{aligned} L_2 &= [L_s, L_{s'}]_\theta = [\mathcal{BMO}(\mathbb{R},\mathcal{M}), \mathcal{H}_1(\mathbb{R},\mathcal{M})]_{\frac{1}{s}}, \mathcal{BMO}(\mathbb{R},\mathcal{M}), \mathcal{H}_1(\mathbb{R},\mathcal{M})]_{\frac{1}{s'}}]_\theta \\ &= [\mathcal{BMO}(\mathbb{R},\mathcal{M}), \mathcal{H}_1(\mathbb{R},\mathcal{M})]_\theta. \end{aligned}$$

□

2.4 Comparison with Mei's results

We denote the column Hardy space defined in [34] through operator-valued Lusin square function by $H_p^c(\mathbb{R},\mathcal{M})$ and the column bounded mean oscillation space appeared in the matrix-valued harmonic analysis by $BMO^c(\mathbb{R},\mathcal{M})$ (see e.g. [34]). We have the following result.

Theorem 4.1. *We have*

$$\mathcal{BMO}^c(\mathbb{R},\mathcal{M}) = BMO^c(\mathbb{R},\mathcal{M})$$

with equivalent norms. Similar results holds for the row spaces. Consequently, $\mathcal{BMO}(\mathbb{R},\mathcal{M}) = \mathcal{BMO}(\mathbb{R},\mathcal{M})$ *with equivalent norms.*

The theorem can be easily seen from the corresponding $BMO(\mathbb{R}, H)$-spaces. However, we can exploit the idea of [15] to prove our $\mathcal{BMO}^c(\mathbb{R},\mathcal{M})$ also coincide with that defined by the mean oscillation $BMO(\mathbb{R}, H)$.

Proof. $\mathcal{BMO}^c(\mathbb{R},\mathcal{M}) \subset BMO^c(\mathbb{R},\mathcal{M})$. Let $\varphi \in \mathcal{BMO}_c(\mathbb{R},\mathcal{M})$. As in the beginning of the proof of Theorem 1.2 in [15], fix a finite interval $I \subset \mathbb{R}$, and consider the collections of dyadic intervals

(1) $\mathcal{D}_1 := \{J \in \mathcal{D}; 2|J| > |I|\}$'

(2) $\mathcal{D}_2 := \{J \in \mathcal{D}; 2|J| \leq |I|, 2J \cap 2I = \emptyset\}$,

(3) $\mathcal{D}_3 := \{J \in \mathcal{D}; 2|J| \leq |I|, 2J \cap 2I \neq \emptyset\}$.

Let $a_J = \langle \varphi, \omega_J \rangle$, then we have a priori formal series

$$\varphi_1(x) = \sum_{J \in \mathcal{D}_1} a_J[\omega_J(x) - \omega_J(c_I)], \varphi_i(x) = \sum_{J \in \mathcal{D}_i} a_J \omega_J(x), i = 2, 3,$$

where c_I is the center of the interval I. Denote $\varphi_I = \varphi_1 + \varphi_2 + \varphi_3$, by a similar discussion in [15], we only need to prove:

$$\|\frac{1}{|I|}\int_I |\varphi_I(x)|^2 dx\|_{\mathcal{M}} < \infty.$$

By scaling we can assume:

$$\sup_I \frac{1}{|I|}\|\sum_{J\subset I} |a_J|^2\| = 1.$$

Then we have the obvious bound for individual terms $\|a_J\| \leq |J|^{\frac{1}{2}}$.

Estimates for φ_1:

$$\begin{aligned}
\|\frac{1}{|I|}\int_I |\varphi_1(x)|^2 dx\| &\leq \frac{1}{|I|}(\sum_{J\in\mathcal{D}_1} \|a_J\||\omega_J(x) - \omega_J(c_I)|)^2 dx \\
&\leq c\frac{1}{|I|}\int_I [\sum_{J\in\mathcal{D}_1} |J|^{\frac{1}{2}}|I||J|^{-\frac{3}{2}}(1 + \frac{dist(I,J)}{|J|})^{-2}]^2 dx \\
&= c[\sum_{j=0}^{\infty}\sum_{|J|\in(2^{j-1},2^j]|I|} |I||J|^{-1}(1 + \frac{dist(I,J)}{|J|})^{-2}]^2 < \infty.
\end{aligned}$$

Estimates for φ_2:

$$\begin{aligned}
\|\frac{1}{|I|}\int_I |\varphi_2(x)|^2 dx\| &\leq \frac{1}{|I|}\int_I \|\sum_{\mathcal{D}_2} a_J\omega_J(x)\|^2 dx \\
&\leq \frac{1}{|I|}\int_I (\sum_{\mathcal{D}_2} \|a_J\||\omega_J(x)|)^2 dx \\
&\leq c\frac{1}{|I|}\int_I [\sum_{\mathcal{D}_2} |J|^{\frac{1}{2}}|J|^{-\frac{1}{2}}(\frac{dist(I,J)}{|J|})^{-2}]^2 dx \\
&= c[\sum_{j=1}^{\infty}\sum_{|J|\in(2^{-j-1},2^{-j})|I|, dist(I,J)>2^{-1}|I|} (\frac{dist(I,J)}{|J|})^{-2}]^2 < \infty.
\end{aligned}$$

Estimates for φ_3:

$$\|\frac{1}{|I|}\int_I |\varphi_3(x)|^2 dx\| \leq \frac{1}{|I|}\|\sum_{J\in\mathcal{D}_3} |a_J|^2\| \leq \frac{1}{|I|}\|\sum_{J\subset 4I} |a_J|^2\| < \infty$$

Hence we deduce that:

$$\|\int_I |\varphi_I(x)|^2 dx\|_{\mathcal{M}} \leq c\sum_{i=1}^{3}\|\int_I |\varphi_i(x)|^2 dx\|_{\mathcal{M}} \leq c|I|$$

Now we turn to the proof of inverse direction $BMO^c(\mathbb{R},\mathcal{M}) \subset \mathcal{BMO}^c(\mathbb{R},\mathcal{M})$. Let $\varphi \in BMO^c(\mathbb{R},\mathcal{M})$. The proof is very similar to that of Lemma 4.1 in Mei's work [34]. For any dyadic interval $I \subset \mathbb{R}$, write $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, where $\varphi_1 = (\varphi - \varphi_{2I})\chi_{2I}, \varphi_2 = (\varphi - \varphi_{2I})\chi_{2I^c}, \varphi_3 = \varphi_{2I}$.

Thus

$$\sum_{J\subset I} |\langle \varphi, \omega_J\rangle|^2 \le 2(\sum_{J\subset I} |\langle \varphi_1, \omega_J\rangle|^2 + \sum_{J\subset I} |\langle \varphi_2, \omega_J\rangle|^2)$$

Estimates for φ_1:

$$\|\sum_{J\subset I} |\langle \varphi_1, \omega_J\rangle|^2\| \le \|\int |\varphi_1(x)|^2 dx\| \le c\|\int_{2I} |\varphi - \varphi_{2I}|^2\| \le c|I|$$

Estimates for φ_2:

$$\begin{aligned}
\|\sum_{J\subset I} |\langle \varphi_2, \omega_J\rangle|^2\| &= \|\sum_{J\subset I} |\sum_{k=1}^{\infty} \int_{2^{k+1}I/2^k I} \varphi_2 \omega_J dx|^2\| \\
&\le \|\sum_{J\subset I} (\sum_{k=1}^{\infty} \frac{1}{2^{2k}} \int_{2^{k+1}I/2^k I} |\varphi_2|^2)(\sum_{k=1}^{\infty} 2^{2k} \int_{2^{k+1}I/2^k I} |\omega_J|^2)\| \\
&\le c(\sum_{k=1}^{\infty} \frac{1}{2^{2k}} \|\int_{2^{k+1}I} |\varphi - \varphi_{2I}|^2\|) \\
&\qquad (\sum_{J\subset I} \sum_{k=1}^{\infty} 2^{2k} \int_{2^{k+1}I/2^k I} |\omega_J|^2) \\
&\le c|I| \|\varphi\|^2_{\mathcal{BMO}_c} \sum_{j=0}^{\infty} 2^j \sum_{k=1}^{\infty} \int_{2^{k+1}I/2^k I} 2^{2k} \frac{|2^{-j}I|^3}{|2^k I|^4} \\
&\le c|I|
\end{aligned}$$

Therefore $\|\sum_{J\subset I} |\langle \varphi, \omega_J\rangle|^2\| \le c|I|$, which completes our proof. □

Combined with Theorem 2.3 and Theorem 3.2, we have the following corollary

Corollary 4.2. *For $1 \le p < \infty$, we have*

$$\mathcal{H}_p^c(\mathbb{R},\mathcal{M}) = H_p^c(\mathbb{R},\mathcal{M}).$$

Similar results hold for $\mathcal{H}_p^r$ and H_p^r, and $\mathcal{H}_p$ and H_p.

If $\mathcal{M} = \mathbb{C}$, $\mathcal{H}_1(\mathbb{R},\mathbb{C})$ is just the usual Hardy space $H_1(\mathbb{R})$ on $\mathbb{R}$. $H_1(\mathbb{R})$ also has the following characterization:

$$H_1(\mathbb{R}) = \{f \in L_1(\mathbb{R}) : H(f) \in L_1(\mathbb{R})\},$$

where H is the Hilbert transform. For any $f \in H_1(\mathbb{R})$,

$$\|f\|_{H_1(\mathbb{R})} \approx \|f\|_{L_1(\mathbb{R})} + \|H(f)\|_{L_1(\mathbb{R})}.$$

Thus $H_1(\mathbb{R})$ can be viewed as a subspace of $L_1(\mathbb{R}) \oplus_1 L_1(\mathbb{R})$. The latter direct sum has its natural operator structure as an L_1 space. This induces an operator space structure on $H_1(\mathbb{R})$. Although $(w_I)_{I\in\mathcal{D}}$ is a unconditional basis of $H_1(\mathbb{R})$, Ricard [54] (see also [55]) proved that $H_1(\mathbb{R})$ does not have complete unconditional basis. However, in noncommutative analysis, one can introduce another natural operator space structure on $H_1(\mathbb{R})$ as follows: $S_1(H_1(\mathbb{R})) = \mathcal{H}_1(\mathbb{R}, B(\ell_2))$, where S_1 is the trace class on ℓ_2. Then we have the following result. Note that Ricard [55] obtained a similar result using Hilbert space techniques.

Corollary 4.3. *The complete orthogonal systems $(w_I)_{I\in\mathcal{D}}$ of $L_2(\mathbb{R})$ is a completely unconditional basis for $H_1(\mathbb{R})$ if we define the operator space structure imposed on $H_1(\mathbb{R})$ by $\mathcal{S}_1(H_1(\mathbb{R})) = \mathcal{H}_1(\mathbb{R}, B(\ell_2))$.*

Proof. Fix a finite subset $\mathcal{I} \subset \mathcal{D}$. Let $T_\varepsilon f \doteq \sum_{I\in\mathcal{I}} \varepsilon_I \langle f, w_I\rangle w_I$, where $\varepsilon_I = \pm 1$. By the definition of $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$, the orthogonality of $(w_I)_{I\in\mathcal{D}}$ yields immediately that

$$\begin{aligned}
\|T_\varepsilon f\|_{\mathcal{H}_1^c} &= \Big\| \Big(\sum_{I\in\mathcal{I}} \frac{|\langle f, w_I\rangle|^2}{|I|} \mathbb{1}_I(x) \Big)^{\frac{1}{2}} \Big\|_{L_1(\mathcal{N})} \\
&\leq \Big\| \Big(\sum_{I\in\mathcal{D}} \frac{|\langle f, w_I\rangle|^2}{|I|} \mathbb{1}_I(x) \Big)^{\frac{1}{2}} \Big\|_{L_1(\mathcal{N})} = \|f\|_{\mathcal{H}_1^c}
\end{aligned}$$

Similarly, the above inequality holds for $\mathcal{H}_1^r(\mathbb{R}, \mathcal{M})$. Now, let $f \in \mathcal{H}_1(\mathbb{R}, \mathcal{M})$, then for any $\epsilon > 0$, there exists a decomposition $f = g + h$ such that

$$\|g\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{M})} + \|h\|_{\mathcal{H}_1^r(\mathbb{R},\mathcal{M})} \leq \|f\|_{\mathcal{H}_1(\mathbb{R},\mathcal{M})} + \epsilon.$$

Therefore

$$\begin{aligned}
\|T_\varepsilon f\|_{\mathcal{H}_1(\mathbb{R},\mathcal{M})} &\leq \|T_\varepsilon g\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{M})} + \|T_\varepsilon h\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{M})} \\
&\leq \|g\|_{\mathcal{H}_1^c(\mathbb{R},\mathcal{M})} + \|h\|_{\mathcal{H}_1^r(\mathbb{R},\mathcal{M})} \leq \|f\|_{\mathcal{H}_1(\mathbb{R},\mathcal{M})} + \epsilon.
\end{aligned}$$

Let $\epsilon \to 0$, we get the result. $\square$

Chapter 3

Calderón-Zygmund operators associated to matrix-valued kernels

Introduction

A *semicommutative* CZO has the formal expression

$$Tf(x) \sim \int_{\mathbb{R}^n} k(x,y)(f(y))\,dy,$$

where the kernel acts linearly on the matrix-valued function $f = (f_{ij})$ and satisfies standard size/smoothness Calderón-Zygmund type conditions. This is the operator model for quite a number of problems which have attracted some attention in recent years, including matrix-valued paraproducts, operator-valued Calderón-Zygmund theory or Fourier multipliers on group von Neumann algebras, see [18, 20, 33, 41, 43] and the references therein. To be more precise, let $\mathcal{B}(\ell_2)$ stand for the matrix algebra of bounded linear operators on ℓ_2. Consider the algebra formed by essentially bounded functions $f : \mathbb{R}^n \to \mathcal{B}(\ell_2)$. Its weak operator closure is a von Neumann algebra $\mathcal{A}$ and as such we may construct noncommutative L_p spaces over it. Let us highlight a few significant examples:

- **Scalar kernels.** $k(x,y) \in \mathbb{C}$ and

$$k(x,y)(f(y)) \ = \ \Big(k(x,y)f_{ij}(y)\Big).$$

- **Schur product actions.** $k(x,y) \in \mathcal{B}(\ell_2)$ and

$$k(x,y)(f(y)) \ = \ \Big(k_{ij}(x,y)f_{ij}(y)\Big).$$

- **Fully noncommutative model.** $k(x,y) \in \mathcal{B}(\ell_2)\bar{\otimes}\mathcal{B}(\ell_2)$ and

$$k(x,y)(f(y)) \ = \ \Big(\sum\nolimits_m \mathrm{tr}\big(k_m''(y)f(y)\big)k_m'(x)_{ij}\Big).$$

- **Partial traces, noncommuting kernels.** $k(x,y) \in \mathcal{B}(\ell_2)$ and

$$k(x,y)(f(y)) \ = \ \begin{cases} \Big(\sum_s k_{is}(x,y) f_{sj}(y) \Big), \\ \Big(\sum_s f_{is}(y) k_{sj}(x,y) \Big). \end{cases}$$

Scalar kernels required in [43] a matrix-valued Calderón-Zygmund decomposition in terms of noncommutative martingales and a pseudo-localization principle to control the tails of Tf in the L_2-metric. Hilbert space valued kernels were later considered in [37], see also [34, 48, 51] for previous related results. The second case refers to the Schur matrix product $k(x,y) \bullet f(y)$, considered for the first time in [20] to analyze cross product extensions of classical CZO's. It is instrumental for Hörmander-Mihlin type theorems on Fourier multipliers associated to discrete groups and for Schur multipliers with a Calderón-Zygmund behavior [20, 21]. In the fully noncommutative model, we approximate $k(x,y)$ by a sum of elementary tensors $\sum_m k_m'(x) \otimes k_m''(y)$ and the action is given by

$$Tf(x) \ \sim \ \int_{\mathbb{R}^n} (id \otimes \mathrm{tr}) \Big[k(x,y) \big(\mathbf{1} \otimes f(y) \big) \Big] \, dy.$$

In this case, we regard the space $L_p(\mathcal{A}) = L_p(\mathbb{R}^n; L_p(\mathcal{B}(\ell_2)))$ as a whole. In other words, the noncommutative nature of $L_p(\mathcal{A})$ predominates and the presence of a Euclidean subspace is ignored. That is what happens for purely noncommutative CZO's [22] and justifies the presence of $id \otimes \mathrm{tr}$, to integrate over the full algebra $\mathcal{A}$ and not just over the Euclidean part. The last case refers to matrix-valued kernels acting on f by left/right multiplication, $k(x,y)f(y)$ and $f(y)k(x,y)$. Matrix-valued paraproducts are prominent examples [30, 33, 36, 41, 50]. This is the only case in which the kernel does not commute with f, since the Schur product is abelian and we find $(id \otimes \mathrm{tr})[k(x,y)(\mathbf{1} \otimes f(y))] = (id \otimes \mathrm{tr})[(\mathbf{1} \otimes f(y))k(x,y)]$ by traciality.

Our main goal is to obtain endpoint estimates for CZO's with noncommuting kernels, motivated by a recent estimate from [20] for semicommutative CZO's. If $k(x,y)$ acts linearly on $\mathcal{B}(\ell_2)$ and satisfies the Hörmander smoothness condition in the norm of bounded linear maps on $\mathcal{B}(\ell_2)$, the content of [20, Lemma 1.3] can be summarized as follows

- If T is $L_\infty(\mathcal{B}(\ell_2); L_2^r(\mathbb{R}^n))$-bounded, then $T : L_\infty(\mathcal{A}) \to \mathrm{BMO}_r(\mathcal{A})$,
- If T is $L_\infty(\mathcal{B}(\ell_2); L_2^c(\mathbb{R}^n))$-bounded, then $T : L_\infty(\mathcal{A}) \to \mathrm{BMO}_c(\mathcal{A})$.

Here, the $L_\infty(L_2^c)$-boundedness assumption refers to

$$\Big\| \Big(\int_{\mathbb{R}^n} Tf(x)^* Tf(x) \, dx \Big)^{\frac12} \Big\|_{\mathcal{B}(\ell_2)} \lesssim \Big\| \Big(\int_{\mathbb{R}^n} f(x)^* f(x) \, dx \Big)^{\frac12} \Big\|_{\mathcal{B}(\ell_2)},$$

while the column-BMO norm of a matrix-valued function g is given by

$$\sup_{Q \text{ cube}} \Big\| \Big(\fint_Q \big(g(x) - g_Q \big)^* \big(g(x) - g_Q \big) \, dx \Big)^{\frac12} \Big\|_{\mathcal{B}(\ell_2)}.$$

Taking adjoints —so that the $*$ switches everywhere from left to right— we find $L_\infty(L_2^r)$-boundedness and the row-BMO norm. The noncommutative BMO space $\mathrm{BMO}(\mathcal{A}) = \mathrm{BMO}_r(\mathcal{A}) \cap \mathrm{BMO}_c(\mathcal{A})$ was introduced in [48]. According to [40] it has the expected interpolation behavior in the L_p scale. Thus, standard interpolation and duality arguments show that $T : L_p(\mathcal{A}) \to L_p(\mathcal{A})$ for $1 < p < \infty$ provided the kernel is smooth enough in both variables and T is a normal self-adjoint map satisfying the $L_\infty(L_2^r)$ and $L_\infty(L_2^c)$ boundedness assumptions. In other words, the row/column boundedness conditions essentially play the role of the L_2-boundedness assumption in classical Calderón-Zygmund theory.

Although this certainly works for non-scalar kernels —Schur product actions were used e.g. in [20, Theorem B]— the boundedness assumptions impose nearly commuting conditions on the kernel which are too strong for CZO's associated to noncommuting kernels. Namely, given $k : \mathbb{R}^{2n} \setminus \Delta \to \mathcal{B}(\ell_2)$ smooth and given $x \notin \mathrm{supp}_{\mathbb{R}^n} f$, let us set formally the row/column CZO's

$$T_c f(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\, dy \quad \text{and} \quad T_r f(x) = \int_{\mathbb{R}^n} f(y) k(x,y)\, dy.$$

It is not difficult to construct noncommuting kernels with

i) T_r and T_c are $L_2(\mathcal{A})$-bounded,

ii) T_r and T_c are not $L_p(\mathcal{A})$-bounded for $1 < p \neq 2 < \infty$,

see e.g. [43, Section 6.1] for specific examples. Therefore, the $L_\infty(L_2^r)$ and $L_\infty(L_2^c)$ boundedness assumption is in general too restrictive when kernel and function do not commute. Assume for what follows that T_r and T_c are $L_2(\mathcal{A})$-bounded. We are interested in weakened forms of L_p boundedness and endpoint estimates for these CZO's. A *dyadic noncommuting* CZO will be a $L_2(\mathcal{A})$-bounded pair (T_r, T_c) associated to a noncommuting kernel satisfying one of the following conditions:

a) Perfect dyadic kernels

$$\big\| k(x,y) - k(z,y) \big\|_{\mathcal{B}(\ell_2)} + \big\| k(y,x) - k(y,z) \big\|_{\mathcal{B}(\ell_2)} = 0$$

whenever $x, z \in Q$ and $y \in R$ for some disjoint dyadic cubes Q, R.

b) Cancellative Haar shift operators

$$k(x,y) = \sum_{Q \text{ dyadic}} \ \sum_{\substack{R,S \text{ dyadic} \subset Q \\ \ell(R) = 2^{-r}\ell(Q) \\ \ell(S) = 2^{-s}\ell(S)}} \alpha_{RS}^Q h_R(x) h_S(y),$$

for some fixed $r, s \in \mathbb{Z}_+$ where the $\alpha_{RS}^Q \in \mathcal{B}(\ell_2)$ with $\|\alpha_{RS}^Q\|_{\mathcal{B}(\ell_2)} \leq \frac{\sqrt{|R||S|}}{|Q|}$. Here h_Q refers to any of the $2^n - 1$ Haar functions related to the cube Q.

Perfect dyadic kernels were introduced in [1] and include Haar multipliers, as well as paraproducts and their adjoints. If J_- and J_+ denote the left/right halves of a dyadic interval in $\mathbb{R}$, the standard model for Haar shifts is the dyadic Hilbert transform with kernel $\sum_J (h_{J_-}(y) - h_{J_+}(y)) h_J(x)$. It appeared after Petermichl's crucial result [46], showing the classical Hilbert transform as a certain average of dyadic Hilbert transforms. Hytönen's representation theorem [14] extends this result to arbitrary CZO's. We will write *generic noncommuting* CZO for $L_2(\mathcal{A})$-bounded pairs (T_r, T_c) with a noncommuting kernel satisfying the standard smoothness. Our first significant result is the following.

Theorem A. *The following inequalities hold*:

i) Dyadic noncommuting CZO's. *Given* $f \in L_1(\mathcal{A})$

$$\inf_{f = f_r + f_c} \big\| T_r f_r \big\|_{1,\infty} + \big\| T_c f_c \big\|_{1,\infty} \lesssim \|f\|_1.$$

ii) Generic noncommuting CZO's. *Given* $f \in \mathrm{H}_1(\mathcal{A})$

$$\inf_{f=f_r+f_c} \big\|T_r f_r\big\|_1 + \big\|T_c f_c\big\|_1 \lesssim \|f\|_{\mathrm{H}_1(\mathcal{A})}.$$

The noncommutative forms of $L_{1,\infty}$ and the Hardy space H_1 are well-known in the subject. Nevertheless, they will also be properly defined in the body of the chapter. Our main result is the inequality given in Theorem A i) and their noncommutative generalizations in Theorem C below. As we shall explain in the Appendix, the left/right modular nature of T_r/T_c is essential for the weak type $(1,1)$ estimates, see also Remark 2.5. The following result easily follows from Theorem A by interpolation/duality and it can also be derived from [20]. Nevertheless, it is worth mentioning the L_p inequalities that we find.

Theorem B. *The following inequalities hold for generic noncommuting CZO's*:

i) *If* $1 < p < 2$ *and* $f \in L_p(\mathcal{A})$

$$\inf_{f=f_r+f_c} \big\|T_r f_r\big\|_p + \big\|T_c f_c\big\|_p \lesssim \|f\|_p.$$

ii) *If* $2 < p < \infty$ *and* $f \in L_p(\mathcal{A})$

$$\big\|T_r f\big\|_{\mathrm{H}_p^r(\mathcal{A})} + \big\|T_c f\big\|_{\mathrm{H}_p^c(\mathcal{A})} \lesssim \|f\|_p.$$

iii) *Given* $f \in L_\infty(\mathcal{A})$, *we also have* $\|T_r f\|_{\mathrm{BMO}_r(\mathcal{A})} + \|T_c f\|_{\mathrm{BMO}_c(\mathcal{A})} \lesssim \|f\|_\infty$.

Theorems A and B also hold for other operator-valued functions, replacing $\mathcal{B}(\ell_2)$ by any semifinite von Neumann algebra $\mathcal{M}$. Our proof will be written in this framework. Let us now consider a weak-$*$ dense filtration $\Sigma_{\mathcal{A}} = (\mathcal{A}_n)_{n\geq 1}$ of von Neumann subalgebras of an arbitrary semifinite von Neumann algebra $\mathcal{A}$. In the following result, we will consider two kind of operators in $L_p(\mathcal{A})$:

a) Noncommuting martingale transforms

$$M_\xi^r f = \sum_{k\geq 1} \Delta_k(f)\xi_{k-1} \quad \text{and} \quad M_\xi^c f = \sum_{k\geq 1} \xi_{k-1}\Delta_k(f).$$

b) Paraproducts with noncommuting symbol

$$\Pi_\rho^r(f) = \sum_{k\geq 1} \mathsf{E}_{k-1}(f)\Delta_k(\rho) \quad \text{and} \quad \Pi_\rho^c(f) = \sum_{k\geq 1} \Delta_k(\rho)\mathsf{E}_{k-1}(f).$$

Here Δ_k denotes the martingale difference operator $\mathsf{E}_k - \mathsf{E}_{k-1}$ and $\xi_k \in \mathcal{A}_k$ is an adapted sequence. Of course, the symbols ξ and ρ do not necessarily commute with the function. Randrianantoanina considered in [51] noncommutative martingale transforms with commuting coefficients. As for paraproducts with noncommuting symbols, Mei studied the L_p-boundedness for $p > 2$ and regular filtrations in [33] and also analyzed in [36] the case $p < 2$ in the dyadic matrix-valued case under a strong BMO condition of the symbol. Our theorem below goes beyond these results, see also [37] for related results.

Theorem C. *Consider the pairs*:

i) Martingale transforms (M_ξ^r, M_ξ^c), *with* $\sup_k \|\xi_k\|_{\mathcal{M}} < \infty$.

ii) Martingale paraproducts (Π_ρ^r, Π_ρ^c), *with* $\Pi_\rho^{r/c}$ $L_2(\mathcal{A})$*-bounded.*

If $\Sigma_{\mathcal{A}}$ *is regular, we obtain weak type* $(1,1)$ *inequalities like in Theorem* Ai) *for martingale transforms and paraproducts . The estimates in Theorems* Aii) *and* B *also hold for both families and for arbitrary filtrations* $\Sigma_{\mathcal{A}}$*. Moreover, the martingale paraproducts* Π_ρ^r *and* Π_ρ^c *are* L_p*-bounded for* $2 < p < \infty$ *and* $L_\infty \to \mathrm{BMO}$*.*

In the case of martingale transforms, there are also examples of noncommuting kernels failing L_p-boundedness for $p \neq 2$. Hence, our results recover those in [51, 52] and are in some sense sharp, providing appropriate substitutes for noncommuting coefficients. Our result for paraproducts goes beyond [33, Theorem 1.2] in two aspects. First, our estimates for $p > 2$ hold for arbitrary martingales, not just for regular ones. Second, we give a partial answer to Mei's question in [33] after the proof of Theorem 1.2 for the case $p < 2$ and also for the weak type $(1,1)$ estimates. The chapter is organized following the order in the Introduction. We include an Appendix at the end with further comments and open problems. Along the chapter we shall assume some familiarity with basic notions from noncommutative integration. The content of [43, Section 1] is enough for our purposes, more can be found in [29, 49, 56].

3.1 Calderón-Zygmund decomposition

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. Consider the algebra of essentially bounded functions $\mathbb{R}^n \to \mathcal{M}$ equipped with the n.s.f. trace

$$\varphi(f) = \int_{\mathbb{R}^n} \tau(f(x))\, dx.$$

Its weak-operator closure is a von Neumann algebra $\mathcal{A}$. If $1 \leq p \leq \infty$, we write $L_p(\mathcal{M})$ and $L_p(\mathcal{A})$ for the noncommutative L_p spaces associated to the pairs $(\mathcal{M}, \tau)$ and $(\mathcal{A}, \varphi)$. The lattices of projections are written $\mathcal{M}_\pi$ and $\mathcal{A}_\pi$, while $\mathbf{1}_{\mathcal{M}}$ and $\mathbf{1}_{\mathcal{A}}$ stand for the unit elements. The set of dyadic cubes in $\mathbb{R}^n$ is denoted by $\mathcal{Q}$ and we use $\mathcal{Q}_k$ for the k-th generation, formed by cubes Q with side length $\ell(Q) = 2^{-k}$. If $f : \mathbb{R}^n \to \mathcal{M}$ is integrable on $Q \in \mathcal{Q}$, we set the average

$$f_Q = \frac{1}{|Q|} \int_Q f(y)\, dy.$$

Let us write $(\mathsf{E}_k)_{k \in \mathbb{Z}}$ for the family of conditional expectations associated to the classical dyadic filtration on $\mathbb{R}^n$. E_k will also stand for the tensor product $\mathsf{E}_k \otimes id_{\mathcal{M}}$ acting on $\mathcal{A}$. If $1 \leq p \leq \infty$ and $f \in L_p(\mathcal{A})$

$$\begin{array}{rclcl} \mathsf{E}_k(f) & = & f_k & = & \displaystyle\sum_{Q \in \mathcal{Q}_k} f_Q 1_Q, \\ \Delta_k(f) & = & df_k & = & \displaystyle\sum_{Q \in \mathcal{Q}_k} (f_Q - f_{\widehat{Q}}) 1_Q, \end{array}$$

where $\widehat{Q}$ denotes the dyadic parent of Q. We will write $(\mathcal{A}_k)_{k \in \mathbb{Z}}$ for the filtration $\mathcal{A}_k = \mathsf{E}_k(\mathcal{A})$. The noncommutative weak L_1-space, denoted by $L_{1,\infty}(\mathcal{A})$, is the set of all φ-measurable operators f for which $\|f\|_{1,\infty} = \sup_{\lambda > 0} \lambda \varphi\{|f| > \lambda\} < \infty$, see [8] for a more in depth discussion. In this case, we write $\varphi\{|f| > \lambda\}$ to denote the trace of the spectral projection of $|f|$ associated to the interval (λ, ∞). We find this terminology more intuitive, since it is reminiscent of the classical one. The space $L_{1,\infty}(\mathcal{A})$ is a quasi-Banach space and satisfies the quasi-triangle inequality below which will be used with no further reference

$$\lambda\,\varphi\Big\{|f_1+f_2|>\lambda\Big\} \leq \lambda\,\varphi\Big\{|f_1|>\lambda/2\Big\} + \lambda\,\varphi\Big\{|f_2|>\lambda/2\Big\}.$$

Let us consider the dense subspace

$$\mathcal{A}_{c,+} = L_1(\mathcal{A}) \cap \Big\{ f:\mathbb{R}^n\to\mathcal{M} \;\Big|\; f\in\mathcal{A}_+,\ \mathrm{supp}_{\mathbb{R}^n} f \ \text{ is compact}\Big\} \subset L_1^+(\mathcal{A}).$$

Here $\mathrm{supp}_{\mathbb{R}^n}$ means the support of f as a vector-valued function in $\mathbb{R}^n$. In other words, we have $\mathrm{supp}_{\mathbb{R}^n} f = \mathrm{supp}\,\|f\|_{\mathcal{M}}$. We employ this terminology to distinguish from $\mathrm{supp}\, f$, the support of f as an operator in $\mathcal{A}$. Any function $f\in\mathcal{A}_{c,+}$ gives rise to a martingale $(f_k)_{k\in\mathbb{Z}}$ with respect to the dyadic filtration. Moreover, it is clear that given $f\in\mathcal{A}_{c,+}$ and $\lambda>0$, there must exist $m_\lambda(f)\in\mathbb{Z}$ so that $0\leq f_k\leq\lambda$ for all $k\leq m_\lambda(f)$. The noncommutative analogue of the weak type $(1,1)$ boundedness of Doob's maximal function is due to Cuculescu. Here we state it in the context of operator-valued functions from $\mathcal{A}$.

Cuculescu's construction [6]. Let $f\in\mathcal{A}_{c,+}$ and consider the corresponding martingale $(f_k)_{k\in\mathbb{Z}}$ relative to the filtration $(\mathcal{A}_k)_{k\in\mathbb{Z}}$. Given $\lambda\in\mathbb{R}_+$, there exists a decreasing sequence of projections $(q_k(\lambda))_{k\in\mathbb{Z}}$ in $\mathcal{A}$ satisfying

i) $q_k(\lambda)$ commutes with $q_{k-1}(\lambda)f_kq_{k-1}(\lambda)$ for each k,

ii) $q_k(\lambda)$ belongs to $\mathcal{A}_k$ for each k and $q_k(\lambda)f_kq_k(\lambda)\leq\lambda q_k(\lambda)$,

iii) The following estimate holds

$$\varphi\Big(\mathbf{1}_{\mathcal{A}} - \bigwedge_{k\in\mathbb{Z}} q_k(\lambda)\Big) \;\leq\; \frac{1}{\lambda}\sup_{k\in\mathbb{Z}}\|f_k\|_1 \;=\; \frac{1}{\lambda}\|f\|_1.$$

Explicitly, take $q_k(\lambda)=\chi_{(0,\lambda]}(q_{k-1}(\lambda)f_kq_{k-1}(\lambda))$ with $q_k(\lambda)=\mathbf{1}_{\mathcal{A}}$ for $k\leq m_\lambda(f)$.

Given $f\in\mathcal{A}_{c,+}$, consider the Cuculescu's sequence $(q_k(\lambda))_{k\in\mathbb{Z}}$ associated to (f,λ) for a given $\lambda>0$. Since λ will be fixed most of the time, we will shorten the notation by q_k and only write $q_k(\lambda)$ when needed. Define the sequence $(p_k)_{k\in\mathbb{Z}}$ of disjoint projections $p_k=q_{k-1}-q_k$, so that

$$\sum_{k\in\mathbb{Z}} p_k = \mathbf{1}_{\mathcal{A}} - q \quad\text{with}\quad q=\bigwedge_{k\in\mathbb{Z}} q_k.$$

Calderón-Zygmund decomposition [43]. Given $f\in\mathcal{A}_{c,+}$ and $\lambda>0$, we may decompose $f=g_d+g_{\mathit{off}}+b_d+b_{\mathit{off}}$ as the sum of four operators defined in terms of the Cuculescu's construction as follows

$$\begin{aligned}
g_d &= qfq+\sum_{k\in\mathbb{Z}} p_kf_kp_k,\\
b_d &= \sum_{k\in\mathbb{Z}} p_k(f-f_k)p_k,\\
b_{\mathit{off}} &= \sum_{i\neq j} p_i(f-f_{i\vee j})p_j,\\
g_{\mathit{off}} &= \sum_{i\neq j} p_if_{i\vee j}p_j \;+\; qf(\mathbf{1}_{\mathcal{A}}-q)+(\mathbf{1}_{\mathcal{A}}-q)fq.
\end{aligned}$$

Moreover, we have the diagonal estimates

$$\Big\|qfq+\sum_{k\in\mathbb{Z}}p_kf_kp_k\Big\|_2^2\leq 2^n\lambda\,\|f\|_1 \quad \text{and} \quad \sum_{k\in\mathbb{Z}}\big\|p_k(f-f_k)p_k\big\|_1\leq 2\,\|f\|_1.$$

The expression below for g_{off} will be also instrumental

$$g_{off}=\sum_{s=1}^{\infty}\sum_{k=m_\lambda+1}^{\infty}p_kdf_{k+s}q_{k+s-1}+q_{k+s-1}df_{k+s}p_k=\sum_{s=1}^{\infty}\sum_{k=m_\lambda+1}^{\infty}g_{k,s}=\sum_{s=1}^{\infty}g_{(s)}.$$

3.2 Proof of Theorems A and B

The key result of this chapter is Theorem A, since the remaining theorems follow from it or by using analog ideas. We begin with the proof of the weak type estimates for perfect dyadic CZO's and then make the necessary adjustments to make it work for Haar shift operators. The proof of Theorem Aii) will require to recall some recent results on square function and atomic Hardy spaces.

3.2.1 Perfect dyadic CZO's

To the best of our knowledge, the notion of perfect dyadic Calderón-Zygmund operator was rigorously defined for the first time in [1] by Auscher, Hofmann, Muscalu, Tao and Thiele. Accordingly, we define a *perfect dyadic* CZO *with noncommuting kernel* as a pair (T_r, T_c) formally given by

$$\begin{aligned} T_rf(x) &\sim \int_{\mathbb{R}^n}k(x,y)f(y)\,dy,\\ T_cf(x) &\sim \int_{\mathbb{R}^n}f(y)k(x,y)\,dy, \end{aligned}$$

with an $\mathcal{M}$-valued kernel satisfying the perfect dyadic conditions

$$\big\|k(x,y)-k(z,y)\big\|_{\mathcal{M}}+\big\|k(y,x)-k(y,z)\big\|_{\mathcal{M}}=0$$

whenever $x,z\in Q$ and $y\in R$ for some disjoint dyadic cubes Q,R. Alternatively, we may think of perfect dyadic kernels $k:\mathbb{R}^{2n}\setminus\Delta\to\mathcal{M}$ as those which are constant on $2n$-cubes of the form $Q\times R$, where Q,R are distinct dyadic cubes in $\mathbb{R}^n$ with the same side length and sharing the same dyadic parent. Classical perfect dyadic CZO's include Haar multipliers/martingale transforms and dyadic paraproducts. In other words, operators of the following form

$$\begin{aligned} H_\xi f(x) &= \int_{\mathbb{R}^n}\Big(\sum_{Q\in\mathcal{Q}}\frac{\xi(\widehat{Q})}{|Q|}1_Q(x)(1_Q-2^{-n}1_{\widehat{Q}})(y)\Big)f(y)\,dy,\\ \Pi_\rho f(x) &= \int_{\mathbb{R}^n}\Big(\sum_{Q\in\mathcal{Q}}\frac{1}{|Q|}(\rho_Q-\rho_{\widehat{Q}})1_Q(x)2^{-n}1_{\widehat{Q}}(y)\Big)f(y)\,dy, \end{aligned}$$

with $\sup_Q|\xi(Q)|<\infty$ and $\rho:\mathbb{R}^n\to\mathbb{C}$ in dyadic BMO. Adjoints of paraproducts are also perfect dyadic. In the noncommuting setting, the coefficients $\xi(Q)$ and the symbol ρ become operators in $\mathcal{M}$ and an $\mathcal{M}$-valued function respectively which do not commute a priori with $f\in L_p(\mathcal{A})$. Nevertheless, the perfect dyadic condition for the kernel is still satisfied in these cases.

Proof of Theorem Ai) — Perfect dyadic CZO's. Splitting f as a sum of four positive operators and by density of $\mathcal{A}_{c,+}$ in the positive cone of $L_1(\mathcal{A})$, we may clearly assume that $f \in \mathcal{A}_{c,+}$. A well-known lack of Cuculescu's construction is that we do not necessarily have $q_k(\lambda_1) \leq q_k(\lambda_2)$ for $\lambda_1 \leq \lambda_2$. This is typically solved restricting our attention to lacunary values for λ. Define

$$\pi_{j,k} = \bigwedge_{s \geq j} q_k(2^s) - \bigwedge_{s \geq j-1} q_k(2^s) \quad \text{for} \quad j,k \in \mathbb{Z}.$$

We have $\sum_j \pi_{j,k} \overset{\mathrm{SOT}}{=} \mathbf{1}_{\mathcal{A}} - \psi_k$, where

$$\psi_k = \bigwedge_{s \in \mathbb{Z}} q_k(2^s).$$

Observe that $\psi_k df_k = df_k \psi_k = 0$ for $k \in \mathbb{Z}$. Indeed, we have

$$\begin{aligned} \|\psi_k df_k\|_{\mathcal{A}} &\leq \|\psi_k f_k^{\frac12}\|_{\mathcal{A}} \|f_k\|_{\mathcal{A}}^{\frac12} + \|\psi_k f_{k-1}^{\frac12}\|_{\mathcal{A}} \|f_{k-1}\|_{\mathcal{A}}^{\frac12} \\ &= \|\psi_k f_k \psi_k\|_{\mathcal{A}}^{\frac12} \|f_k\|_{\mathcal{A}}^{\frac12} + \|\psi_k f_{k-1} \psi_k\|_{\mathcal{A}}^{\frac12} \|f_{k-1}\|_{\mathcal{A}}^{\frac12} \leq \lim_{s \to -\infty} 2^{1+\frac{s}{2}} \|f\|_{\mathcal{A}}^{\frac12}. \end{aligned}$$

In particular, we find $f = \sum_k (\mathbf{1}_{\mathcal{A}} - \psi_{k-1}) df_k (\mathbf{1}_{\mathcal{A}} - \psi_{k-1})$ and set $f = f_r + f_c$ with

$$\begin{aligned} f_r &= \sum_{k \in \mathbb{Z}} \mathsf{LT}_{k-1}(df_k) = \sum_{k \in \mathbb{Z}} \Big(\sum_{i > j} \pi_{i,k-1} df_k \pi_{j,k-1} \Big), \\ f_c &= \sum_{k \in \mathbb{Z}} \mathsf{UT}_{k-1}(df_k) = \sum_{k \in \mathbb{Z}} \Big(\sum_{i \leq j} \pi_{i,k-1} df_k \pi_{j,k-1} \Big). \end{aligned}$$

This is the decomposition we will use for any perfect dyadic CZO. Given such an operator $T = (T_r, T_c)$ and $\lambda > 0$, the goal is to show that there exists an absolute constant c_0 so that $\lambda\varphi\{|T_r f_r| > \lambda\} + \lambda\varphi\{|T_c f_c| > \lambda\} \leq c_0 \|f\|_1$ for any $f \in \mathcal{A}_{c,+}$ and any $\lambda > 0$. By symmetry in the argument, we will just prove the inequality for $T_c f_c$. Moreover, replacing c_0 by $2c_0$ we may also assume that $\lambda = 2^\ell$ for some $\ell \in \mathbb{Z}$. Having fixed the value of λ, we may consider the Calderón-Zygmund decomposition $f = g_d + g_{\mathit{off}} + b_d + b_{\mathit{off}}$ and set

$$\begin{aligned} g_d^c &= \sum_{k \in \mathbb{Z}} \mathsf{UT}_{k-1}\big(\Delta_k(g_d)\big), \qquad & g_{\mathit{off}}^c &= \sum_{k \in \mathbb{Z}} \mathsf{UT}_{k-1}\big(\Delta_k(g_{\mathit{off}})\big), \\ b_d^c &= \sum_{k \in \mathbb{Z}} \mathsf{UT}_{k-1}\big(\Delta_k(b_d)\big), \qquad & b_{\mathit{off}}^c &= \sum_{k \in \mathbb{Z}} \mathsf{UT}_{k-1}\big(\Delta_k(b_{\mathit{off}})\big). \end{aligned}$$

By the quasi-triangle inequality it suffices to show

$$\lambda \Big[\varphi\Big\{ |T_c g_d^c| > \lambda \Big\} + \varphi\Big\{ |T_c b_d^c| > \lambda \Big\} + \varphi\Big\{ |T_c g_{\mathit{off}}^c| > \lambda \Big\} + \varphi\Big\{ |T_c b_{\mathit{off}}^c| > \lambda \Big\} \Big] \lesssim \|f\|_1.$$

The first term is first estimated by Chebychev's inequality in $\mathcal{A}$

$$\lambda\varphi\Big\{ |T_c g_d^c| > \lambda \Big\} \leq \frac{1}{\lambda} \big\| T_c g_d^c \big\|_2^2 \lesssim \frac{1}{\lambda} \|g_d^c\|_2^2.$$

We use that $\mathsf{UT}_{k-1}\big(\Delta_k(g_d)\big)$ are in fact martingale differences, so that

$$\frac{1}{\lambda} \|g_d^c\|_2^2 = \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \big\| \mathsf{UT}_{k-1}\big(\Delta_k(g_d)\big) \big\|_2^2 \leq \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \|\Delta_k(g_d)\|_2^2$$

$$= \frac{1}{\lambda}\Big\| \sum_{k\in\mathbb{Z}} \Delta_k(g_d) \Big\|_2^2 \ = \ \frac{1}{\lambda}\Big\| qfq + \sum_{k\in\mathbb{Z}} p_k f_k p_k \Big\|_2^2 \ \leq \ 2^n \|f\|_1.$$

Indeed, the first inequality above follows from the fact that triangular truncations are contractive in $L_2(\mathcal{A})$ while the last inequality arise from the diagonal estimates in the noncommutative CZ decomposition stated above. To handle the remaining terms, we introduce the projection

$$\widehat{q} \ = \ \bigwedge_{s\geq\ell} q(2^s) \ = \ \bigwedge_{s\geq\ell} \bigwedge_{k\in\mathbb{Z}} q_k(2^s).$$

According to Cuculescu's construction, we find

$$\varphi\big(\mathbf{1}_{\mathcal{A}} - \widehat{q}\big) \ \leq \ \sum_{s\geq\ell} \varphi\big(\mathbf{1}_{\mathcal{A}} - q(2^s)\big) \ \leq \ \sum_{s\geq\ell} \frac{1}{2^s}\|f\|_1 \ = \ \frac{2}{\lambda}\|f\|_1.$$

This reduces our problem to show that

$$\lambda\Big[\varphi\Big\{\big|T_c(b_d^c)\widehat{q}\big| > \lambda\Big\} + \varphi\Big\{\big|T_c(g_{\mathit{off}}^c)\widehat{q}\big| > \lambda\Big\} + \varphi\Big\{\big|T_c(b_{\mathit{off}}^c)\widehat{q}\big| > \lambda\Big\}\Big] \ \lesssim \ \|f\|_1.$$

The perfect dyadic nature of T_c comes now into scene. Indeed, we claim that the three terms $T_c(b_d^c)\widehat{q}, T_c(g_{\mathit{off}}^c)\widehat{q}, T_c(b_{\mathit{off}}^c)\widehat{q}$ vanish whenever T_c is perfect dyadic. This will be enough to conclude the proof. If $Q_k(x)$ is the only cube in $\mathcal{Q}_k$ containing x, we find a.e. x

$$\begin{aligned} T_c(b_d^c)(x)\widehat{q}(x) \ &= \ \sum_{k\in\mathbb{Z}} T_c\big(\mathsf{UT}_{k-1}(\Delta_k(b_d))\big)(x)\,\widehat{q}(x) \\ &= \ \sum_{k\in\mathbb{Z}} T_c\Big(\mathsf{UT}_{k-1}(\Delta_k(b_d))1_{Q_{k-1}(x)}\Big)(x)\,\widehat{q}(x) \\ &+ \ \sum_{k\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{Q}_{k-1}\\ x\notin Q}} \Big(\int_Q k(x,y)\mathsf{UT}_{k-1}(\Delta_k(b_d))(y)\,dy\Big)\,\widehat{q}(x). \end{aligned}$$

The last term on the right vanishes since the term $\mathsf{UT}_{k-1}(\Delta_k(b_d))$ has mean 0 in any $Q \in \mathcal{Q}_{k-1}$, so that we may replace $k(x,y)$ by $k(x,y) - k(x,c_Q)$, which is 0 when $x \notin Q$ by the perfect dyadic cancellation of the kernel. On the other hand, if we define the projection

$$\widehat{q}_{k-1} = \bigwedge_{s\geq\ell} q_{k-1}(2^s),$$

we see that $\widehat{q}(x) = \widehat{q}_{k-1}(x)\widehat{q}(x) = \widehat{q}_{k-1}(y)\widehat{q}(x)$ for any $y \in Q_{k-1}(x)$. This gives

$$T_c(b_d^c)(x)\widehat{q}(x) \ = \ \sum\nolimits_k T_c\Big(\mathsf{UT}_{k-1}(\Delta_k(b_d))\widehat{q}_{k-1}1_{Q_{k-1}(x)}\Big)(x)\,\widehat{q}(x).$$

The exact same argument applies for g_{off}^c and b_{off}^c, so that it suffices to prove

$$\begin{aligned} \mathsf{UT}_{k-1}(\Delta_k(b_d))\,\widehat{q}_{k-1} \ &= \ 0, \\ \mathsf{UT}_{k-1}(\Delta_k(g_{\mathit{off}}))\,\widehat{q}_{k-1} \ &= \ 0, \\ \mathsf{UT}_{k-1}(\Delta_k(b_{\mathit{off}}))\,\widehat{q}_{k-1} \ &= \ 0, \end{aligned}$$

for all $k \in \mathbb{Z}$. In all these cases we will be using the following two key identities

- $\widehat{q}_{k-1}\pi_{i,k-1} = \pi_{j,k-1}\widehat{q}_{k-1} = 0$ for $i, j > \ell$ and $k \in \mathbb{Z}$,

- $\pi_{i,k-1}p_{k-s} = p_{k-s}\pi_{j,k-1} = 0$ for $s \geq 1$, $i, j \leq \ell$ and $k \in \mathbb{Z}$.

The proof is straightforward and left to the reader. It only requires to apply the monotonicity properties of $\bigwedge_{s\geq j} q_k(2^s)$, which increases in j and decreases in k. If we apply the first identity to $\mathsf{UT}_{k-1}(\Delta_k(\gamma))\,\widehat{q}_{k-1}$ for any γ, we get

$$\mathsf{UT}_{k-1}(\Delta_k(\gamma))\,\widehat{q}_{k-1} \ = \ \sum_{i\leq j\leq \ell} \pi_{i,k-1} d\gamma_k \pi_{j,k-1}\widehat{q}_{k-1}.$$

Therefore, if we know that $d\gamma_k = A_k + B_k$ where the left support of A_k and the right support of B_k are dominated by $\sum_{s\geq 1} p_{k-s} = \mathbf{1}_{\mathcal{A}} - q_{k-1}$, then we deduce that $\mathsf{UT}_{k-1}(\Delta_k(\gamma))\,\widehat{q}_{k-1} = 0$. In other words, it suffices to prove that

$$q_{k-1}\Delta_k(\gamma)q_{k-1} \ = \ 0 \quad \text{for} \quad \gamma = b_d, g_{\mathit{off}}, b_{\mathit{off}}.$$

We have

$$\begin{aligned}
\Delta_k(b_d) \ &= \ \sum_j \Delta_k\big(p_j(f - f_j)p_j\big) \\
&= \ \sum_{j<k} p_j(f_k - f_j)p_j - \sum_{j<k-1} p_j(f_{k-1} - f_j)p_j \\
&= \ \sum_{j\leq k-1} p_j df_k p_j \ = \ (\mathbf{1}_{\mathcal{A}} - q_{k-1})\Delta_k(b_d)(\mathbf{1}_{\mathcal{A}} - q_{k-1}).
\end{aligned}$$

To calculate the martingale differences for g_{off}, we invoke the formula

$$g_{\mathit{off}} = \sum_{s=1}^{\infty}\sum_{j\in\mathbb{Z}} p_j df_{j+s} q_{j+s-1} + q_{j+s-1} df_{j+s} p_j$$

given in the statement of the Calderón-Zygmund decomposition. Then we find

$$\begin{aligned}
\Delta_k(g_{\mathit{off}}) \ &= \ \sum_{s=1}^{\infty} p_{k-s} df_k q_{k-1} + q_{k-1} df_k p_{k-s} \\
&= \ (\mathbf{1}_{\mathcal{A}} - q_{k-1}) df_k q_{k-1} + q_{k-1} df_k (\mathbf{1}_{\mathcal{A}} - q_{k-1}).
\end{aligned}$$

Finally, it remains to consider the martingale differences of b_{off}

$$\begin{aligned}
\Delta_k(b_{\mathit{off}}) \ &= \ \sum_{s=1}^{\infty}\sum_{j\in\mathbb{Z}} \Delta_k\big(p_j(f - f_{j+s})p_{j+s} + p_{j+s}(f - f_{j+s})p_j\big) \\
&= \ \sum_{s=1}^{\infty}\sum_{j<k-s} p_j(f_k - f_{j+s})p_{j+s} + p_{j+s}(f_k - f_{j+s})p_j \\
&- \ \sum_{s=1}^{\infty}\sum_{j<k-s-1} p_j(f_{k-1} - f_{j+s})p_{j+s} + p_{j+s}(f_{k-1} - f_{j+s})p_j \\
&= \ \sum_{s=1}^{\infty}\sum_{j<k-s} p_j df_k p_{j+s} + \sum_{s=1}^{\infty}\sum_{j<k-s} p_{j+s} df_k p_j \ = \ A_k + B_k.
\end{aligned}$$

So $q_{k-1}A_k = B_k q_{k-1} = 0$ and $q_{k-1}\Delta_k(\gamma)q_{k-1} = 0$ for $\gamma = b_d, g_{\mathit{off}}, b_{\mathit{off}}$ as desired. $\square$

3.2.2 Haar shift operators

The Haar system has the form

$$h_Q^{\varepsilon}(x) \;=\; \frac{1}{\sqrt{|Q|}} \prod_{j=1}^{n} \big(1_{I_j^-}(x_j) + \varepsilon_j 1_{I_j^+}(x_j) \big)$$

where $Q = I_1 \times I_2 \times \cdots \times I_n \in \mathcal{Q}$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \neq (1,1,\ldots,1)$ with $\varepsilon_j \in \pm 1$. We are using I_j^- and I_j^+ for the left/right halves of the intervals I_j. It yields an orthonormal system in $L_2(\mathbb{R}^n)$ composed of mean zero functions. If we write h_Q for any Haar function of the form h_Q^{ε}, a *noncommuting dyadic shift with complexity* (r,s) has the form

$$\text{Ш}_{\alpha} f(x) \;=\; \sum_{Q \in \mathcal{Q}} A_Q f \;=\; \sum_{Q \in \mathcal{Q}} \sum_{\substack{R,S \text{ dyadic} \subset Q \\ \ell(R) = 2^{-r} \ell(Q) \\ \ell(S) = 2^{-s} \ell(Q)}} \alpha_{RS}^{Q} \langle f, h_S \rangle h_R(x),$$

where $\langle f, h_S \rangle = \int f h_S$ and α_{RS}^Q are operators in $\mathcal{M}$ satisfying $\|\alpha_{RS}^Q\|_{\mathcal{M}} \leq \frac{\sqrt{|R||S|}}{|Q|}$.

Lemma 2.1. *We have* $\|\text{Ш}_{\alpha} f\|_2 \leq \|f\|_2$.

Proof. The argument is standard, observe that

$$\|\text{Ш}_{\alpha} f\|_2^2 \;=\; \sum_{Q,Q'} \sum_{R,R'S,S'} \tau\big(\langle f, h_S \rangle^* \alpha_{RS}^{Q*} \alpha_{R'S'}^{Q'} \langle f, h_{S'} \rangle \big) \int_{\mathbb{R}^n} h_R(y) h_{R'}(y)\, dy.$$

The integral on the right imposes $R = R'$, which in turn gives $Q = Q'$ since Q is the unique r-th ancestor of R and the same happens for (R', Q'). Once we know that $Q = Q'$, we may write

$$\|\text{Ш}_{\alpha} f\|_2^2 \;=\; \sum_{Q \in \mathcal{Q}} \|A_Q f\|_2^2 \;=\; \sum_{Q \in \mathcal{Q}} \Big\| A_Q \Big(\sum_{\substack{S \subset Q \\ \ell(S) = 2^{-s} \ell(Q)}} \langle f, h_S \rangle h_S \Big) \Big\|_2^2.$$

It is worth mentioning that the double use above of h_S always refers to the same choice of h_S^{ε} in both instances. On the other hand, it is easily seen that A_Q is a contractive operator on $L_2(\mathcal{A})$. Indeed, we have

$$\begin{aligned} \|A_Q g\|_2^2 \;&\leq\; \int_{\mathbb{R}^n} \Big[\sum_{R,S} \|\alpha_{R,S}^Q\|_{\mathcal{M}} \Big(\frac{1}{\sqrt{|S|}} \int_S \|g(y)\|_{L_2(\mathcal{M})} dy \Big) \frac{1}{\sqrt{|R|}} 1_R(x) \Big]^2 dx \\ &\leq\; \int_Q \Big(\fint_Q \|g(y)\|_{L_2(\mathcal{M})}\, dy \Big)^2 dx \;\leq\; \fint_Q \|g\|_{L_2(\mathcal{A})}^2\, dx \;=\; \|g\|_{L_2(\mathcal{A})}^2. \end{aligned}$$

This yields

$$\|\text{Ш}_{\alpha} f\|_2^2 \leq \sum_{Q \in \mathcal{Q}} \Big\| \sum_{\substack{S \subset Q \\ \ell(S) = 2^{-s} \ell(Q)}} \langle f, h_S \rangle h_S \Big\|_2^2 \;=\; \Big\| \sum_{Q \in \mathcal{Q}} \langle f, h_Q \rangle h_Q \Big\|_2^2 \;=\; \|f\|_2^2. \qquad \square$$

The next lemma is crucial to analyze Haar shifts and general Calderón-Zygmund operators with noncommuting kernels. We take here the opportunity to slightly modify the argument in [43, Lemma 4.2], which was not entirely correct.

Lemma 2.2. *Given* $s \in \mathbb{Z}_+$, *there exists* $\zeta \in \mathcal{A}_\pi$ *such that*:

i) $\lambda\varphi(\mathbf{1}_{\mathcal{A}} - \zeta) \le 2^{sn}\|f\|_1$,

ii) *If* $Q_0 \in \mathcal{Q}_{k_0}$ *and* $x \in \widehat{Q}_0^s$, *then* $\zeta(x) \le \widehat{q}_{k_0}(y)$ *for all* $y \in Q_0$.

In the second property, we write $\widehat{Q}_0^s$ *for the unique* s*-th dyadic ancestor of* Q_0.

Proof. We have

$$\mathbf{1}_{\mathcal{A}} - \widehat{q}_k = \sum_{j\le k} \big(\widehat{q}_{j-1} - \widehat{q}_j\big) = \sum_{j\le k}\sum_{Q\in\mathcal{Q}_j} \rho_Q \otimes 1_Q = \sum_{Q\in\mathcal{Q}_k}\Big[\sum_{R\supset Q}\rho_R\Big]\otimes 1_Q$$

for some family of projections $\rho_Q \in \mathcal{M}_\pi$. Define

$$\zeta = \bigwedge_{k\in\mathbb{Z}} \zeta_k \quad \text{with} \quad \zeta_k = \mathbf{1}_{\mathcal{A}} - \bigvee_{j\le k}\bigvee_{Q\in\mathcal{Q}_j} \rho_Q 1_{\widehat{Q}^s}.$$

It is clear that the ζ_k's are decreasing in k and we find

$$\begin{aligned}
\lambda\varphi(\mathbf{1}_{\mathcal{A}} - \zeta) &= \lambda \lim_{k\to\infty} \varphi(\mathbf{1}_{\mathcal{A}} - \zeta_k) \\
&\le \lambda \lim_{k\to\infty} \sum_{j\le k}\sum_{Q\in\mathcal{Q}_j} \tau(\rho_Q)|\widehat{Q}^s| \\
&= 2^{sn} \lim_{k\to\infty} \lambda \sum_{j\le k}\sum_{Q\in\mathcal{Q}_j} \varphi(\rho_Q\otimes 1_Q) \\
&= 2^{sn}\lambda\,\varphi\big(\mathbf{1}_{\mathcal{A}} - \widehat{q}\big) = 2^{sn}\lambda \sum_{m\ge \ell} \varphi\big(\mathbf{1}_{\mathcal{A}} - q(2^m)\big) \lesssim 2^{sn}\|f\|_1.
\end{aligned}$$

To prove the second property, it will be useful to observe that $Q_1 \subsetneq Q_2$ implies that $\rho_{Q_1} \perp \rho_{Q_2}$ are orthogonal projections. Indeed, according to the definition of ρ_Q above, we have $\rho_{Q_1}\rho_{Q_2}1_{Q_1} = (\widehat{q}_{j_1-1} - \widehat{q}_{j_1})(\widehat{q}_{j_2-1} - \widehat{q}_{j_2})1_{Q_1} = 0$ for $\ell(Q_1) = 2^{-j_1}$ and $\ell(Q_2) = 2^{-j_2}$. Then, we find

$$\begin{aligned}
\zeta(x) &\le \zeta_{k_0}(x) \\
&= \mathbf{1}_{\mathcal{M}} - \bigvee_{j\le k_0}\bigvee_{Q\in\mathcal{Q}_j} \rho_Q 1_{\widehat{Q}^s}(x) \\
&\le \mathbf{1}_{\mathcal{M}} - \bigvee_{R\supset Q_0} \rho_R = 1_{\mathcal{M}} - \sum_{R\supset Q_0} \rho_R \\
&= \Big(\mathbf{1}_{\mathcal{A}} - \sum_{Q\in\mathcal{Q}_{k_0}} \Big[\sum_{R\supset Q}\rho_R\Big]\otimes 1_Q\Big)(y) = \widehat{q}_{k_0}(y).
\end{aligned}$$

□

Proof of Theorem Ai) — Haar shift operators. As in the perfect dyadic case, we assume $f \in \mathcal{A}_{c,+}$ and decompose $f = f_r + f_c$ in the same way. Once more the argument is row/column symmetric, and we just consider the column part. After fixing $\lambda = 2^\ell$ for some $\ell \in \mathbb{Z}$, we construct the corresponding Calderón-Zygmund decomposition for $f_c = g_d^c + g_{\mathit{off}}^c + b_d^c + b_{\mathit{off}}^c$. According to Lemma 2.1, we may control the term $\mathrm{III}_\alpha(g_d^c)$ in the usual way. Given $\gamma \in \{b_d, g_{\mathit{off}}, b_{\mathit{off}}\}$, the other terms can be decomposed as follows

$$\mathrm{III}_\alpha(\gamma^c) = \sum_{k\in\mathbb{Z}} \mathrm{III}_\alpha\big(\mathsf{UT}_{k-1}(\Delta_k(\gamma))\big)$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}} \sum_{\substack{R,S\subset Q\\ \ell(R)=2^{-r}\ell(Q)\\ \ell(S)=2^{-s}\ell(Q)}} \alpha_{RS}^{Q}\Big(\int_{\mathbb{R}^n} \mathsf{UT}_{k-1}(\Delta_k(\gamma))h_S\,dy\Big)h_R(x)\\
&= \sum_{k\in\mathbb{Z}} \Big[\sum_{\substack{Q\in\mathcal{Q}\\ \ell(Q)\le 2^{-k+1}}} + \sum_{\substack{Q\in\mathcal{Q}\\ \ell(Q)>2^{-k+1}\\ \ell(Q)\le 2^{s-k+1}}} + \sum_{\substack{Q\in\mathcal{Q}\\ \ell(Q)>2^{s-k+1}}} \Big] = A_\gamma + B_\gamma + C_\gamma.
\end{aligned}
$$

We claim that $C_\gamma = 0$. Namely, we have $\ell(S) = 2^{-s}\ell(Q) > 2^{-k+1}$. This means that $\mathsf{E}_{k-1}(h_S) = h_S$ since the Haar functions h_S are constant in the dyadic children of S, whose length sides are greater or equal than $2^{-(k-1)}$. This yields

$$
\begin{aligned}
\int_{\mathbb{R}^n} \mathsf{UT}_{k-1}(\Delta_k(\gamma))h_S\,dy &= \int_{\mathbb{R}^n} \mathsf{E}_{k-1}\big(\mathsf{UT}_{k-1}(\Delta_k(\gamma))h_S\big)\,dy\\
&= \int_{\mathbb{R}^n} \mathsf{E}_{k-1}\big(\mathsf{UT}_{k-1}(\Delta_k(\gamma))\big)h_S\,dy\\
&= \int_{\mathbb{R}^n} \big(\mathsf{UT}_{k-1}(\mathsf{E}_{k-1}\Delta_k(\gamma))h_S\,dy \;=\; 0.
\end{aligned}
$$

To deal with the remaining terms A_γ and B_γ, we invoke the identity $q_{k-1}\Delta_k(\gamma)q_{k-1} = 0$ which was already justified in the perfect dyadic case whenever $\gamma = b_d, g_{\mathit{off}}, b_{\mathit{off}}$. Namely, since $\pi_{i,k-1}(\mathbf{1}_\mathcal{A} - q_{k-1}) = (\mathbf{1}_\mathcal{A} - q_{k-1})\pi_{j,k-1} = 0$ for $i,j \le \ell$, we find

$$
\mathsf{UT}_{k-1}(\Delta_k(\gamma)) = \sum_{i\le j} \pi_{i,k-1}\Delta_k(\gamma)\pi_{j,k-1} = \sum_{\substack{i\le j\\ j>\ell}} \pi_{i,k-1}\Delta_k(\gamma)\pi_{j,k-1}.
$$

Let us now consider the term A_γ, we have

$$
\lambda\,\varphi\big\{|A_\gamma| > \lambda\big\} \le \lambda\,\varphi\big(\mathbf{1}_\mathcal{A} - \widehat{q}\big) + \lambda\,\varphi\Big\{\big|A_\gamma\widehat{q}\big| > \frac{\lambda}{2}\Big\}.
$$

We already know that the first term on the right is dominated by $\|f\|_1$ and

$$
A_\gamma\widehat{q} = \sum_{k\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{Q}\\ \ell(Q)\le 2^{-k+1}}} \sum_{\substack{R,S\subset Q\\ \ell(R)=2^{-r}\ell(Q)\\ \ell(S)=2^{-s}\ell(Q)}} \alpha_{RS}^{Q}\Big(\int_{\mathbb{R}^n} \mathsf{UT}_{k-1}(\Delta_k(\gamma))h_S\,dy\Big)h_R(x)\,\widehat{q}(x).
$$

Given $Q \in \mathcal{Q}$ with $\ell(Q) \le 2^{-k+1}$ let

$$
k_Q \ge k-1 \quad \text{determined by} \quad \ell(Q) = 2^{-k_Q}.
$$

It is clear that $\widehat{q}(x) = \widehat{q}_{k_Q}(x)\widehat{q}(x) = \widehat{q}_{k_Q}(y)\widehat{q}(x) = \widehat{q}_{k-1}(y)\widehat{q}(x)$ whenever x, y belong to Q. However, the presence of $h_R(x), h_S(y)$ implies (unless the corresponding term is 0) that the pair $(x,y) \in R\times S \subset Q\times Q$ so that we may write

$$
A_\gamma\widehat{q} = \sum_{k\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{Q}\\ \ell(Q)\le 2^{-k+1}}} \sum_{\substack{R,S\subset Q\\ \ell(R)=2^{-r}\ell(Q)\\ \ell(S)=2^{-s}\ell(Q)}} \alpha_{RS}^{Q}\Big(\int_{\mathbb{R}^n} \mathsf{UT}_{k-1}(\Delta_k(\gamma))\widehat{q}_{k-1}h_S\,dy\Big)h_R(x)\,\widehat{q}(x).
$$

Therefore, we conclude

$$
\mathsf{UT}_{k-1}(\Delta_k(\gamma))\widehat{q}_{k-1} = \sum_{\substack{i\le j\\ j>\ell}} \pi_{i,k-1}\Delta_k(\gamma)\pi_{j,k-1}\widehat{q}_{k-1} = 0
$$

since $\pi_{j,k-1}\widehat{q}_{k-1} = 0$ when $j > \ell$. This shows that $A_\gamma \widehat{q} = 0$. Let us finally consider the term B_γ. We will follow a similar argument with the projection ζ from Lemma 2.2 instead. Namely, we have

$$\lambda\,\varphi\big\{|B_\gamma| > \lambda\big\} \,\leq\, \lambda\,\varphi\big(\mathbf{1}_{\mathcal{A}} - \zeta\big) \,+\, \lambda\,\varphi\Big\{\big|B_\gamma\zeta\,\big| > \frac{\lambda}{2}\Big\}.$$

According to property i) of Lemma 2.2, it suffices to show that $B_\gamma\zeta = 0$. Now we know that $\ell(Q) \leq 2^{s-k+1}$, so that $k_Q \geq k - s - 1$. Let us now consider the 2^{ns} dyadic cubes T_j having Q as their s-th dyadic ancestor. This gives rise to the identities

$$\zeta(x) \,=\, \zeta_{k_Q+s}(x)\zeta(x) \,=\, \zeta_{k_Q+s}(y)\zeta(x) \,=\, \widehat{q}_{k_Q+s}(z)\zeta(x) \,\,=\,\, \widehat{q}_{k-1}(z)\zeta(x)$$

for $(x, y, z) \in Q\times Q\times T_j$. Indeed, the second identity follows from the fact that $\mathsf{E}_{k_Q}(\zeta_{k_Q+s}) = \zeta_{k_Q+s}$, the third one from the second property in Lemma 2.2 and the last one from the inequality $k_Q \geq k - s - 1$. Hence, given $y \in S \subset Q$ we pick the unique j for which $S = T_j$ and deduce that $\zeta(x) = \widehat{q}_{k-1}(y)\zeta(x)$. Then it yields the identity

$$B_\gamma\zeta \,=\, \sum_{k\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{Q} \\ \ell(Q)>2^{-k+1} \\ \ell(Q)\leq 2^{s-k+1}}} \sum_{\substack{R,S\subset Q \\ \ell(R)=2^{-r}\ell(Q) \\ \ell(S)=2^{-s}\ell(Q)}} \alpha_{RS}^{Q}\Big(\int_{\mathbb{R}^n} \mathsf{UT}_{k-1}(\Delta_k(\gamma))\widehat{q}_{k-1}h_S\,dy\Big)h_R(x)\,\zeta(x).$$

The integrand $\mathsf{UT}_{k-1}(\Delta_k(\gamma))\widehat{q}_{k-1}$ vanishes for the same reason as it did above. □

Remark 2.3. Our constants are $\sim 2^{sn}$ and seem far to be sharp. Unfortunately, the classical argument leading to constants $\sim s$ encounters a major obstacle due to the presence —in the noncommutative setting— of triangular truncations, which are not bounded in L_1. The Appendix below contains more details on this topic.

3.2.3 Noncommuting CZO's

The proofs of Theorems Aii), B and C arise from a careful combination of recent results in the theory of noncommutative Hardy spaces. Let us begin introducing Mei's notion [34] of row and column Hardy spaces for our algebra of operator-valued functions $\mathcal{A}$. In order to distinguish from order Hardy spaces to be introduced below, let us follows Mei's notation and define

$$\mathrm{H}_1(\mathbb{R}^n;\mathcal{M}) \,=\, \mathrm{H}_1^r(\mathbb{R}^n;\mathcal{M}) + \mathrm{H}_1^c(\mathbb{R}^n;\mathcal{M})$$

as the space of functions $f \in L_1(\mathcal{A})$ for which we have

$$\|f\|_{\mathrm{H}_1(\mathbb{R}^n;\mathcal{M})} \,=\, \inf_{f=g+h} \|g\|_{\mathrm{H}_1^r(\mathbb{R}^n;\mathcal{M})} + \|h\|_{\mathrm{H}_1^c(\mathbb{R}^n;\mathcal{M})} < \infty,$$

where the row/column norms are given by

$$\begin{aligned} \|g\|_{\mathrm{H}_1^r(\mathbb{R}^n;\mathcal{M})} &= \Big\|\Big(\int_\Gamma \Big[\frac{\partial\widehat{g}}{\partial t}\frac{\partial\widehat{g}^*}{\partial t} + \sum\nolimits_j \frac{\partial\widehat{g}}{\partial x_j}\frac{\partial\widehat{g}^*}{\partial x_j}\Big](x+\cdot,t)\,\frac{dxdt}{t^{n-1}}\Big)^{\frac12}\Big\|_1, \\ \|h\|_{\mathrm{H}_1^c(\mathbb{R}^n;\mathcal{M})} &= \Big\|\Big(\int_\Gamma \Big[\frac{\partial\widehat{h}^*}{\partial t}\frac{\partial\widehat{h}}{\partial t} + \sum\nolimits_j \frac{\partial\widehat{h}^*}{\partial x_j}\frac{\partial\widehat{h}}{\partial x_j}\Big](x+\cdot,t)\,\frac{dxdt}{t^{n-1}}\Big)^{\frac12}\Big\|_1, \end{aligned}$$

with $\Gamma = \{(x,t) \in \mathbb{R}_+^{n+1} \mid |x| < y\}$ and $\widehat{f}(x,t) = P_t f(x)$ for the Poisson semigroup $(P_t)_{t\geq 0}$. In other words, operator-valued forms of Lusin's square function. We say that $a \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^n))$ is a *column atom* if there exists a cube Q so that

- $\mathrm{supp}_{\mathbb{R}^n}\, a = Q$,
- $\displaystyle\int_Q a(y)\, dy = 0$,
- $\displaystyle \|a\|_{L_1(\mathcal{M}; L_2^c(\mathbb{R}^n))} = \tau\Big[\Big(\int_Q |a(y)|^2\, dy\Big)^{\frac{1}{2}}\Big] \le \frac{1}{\sqrt{|Q|}}$.

According to [34, Theorem 2.8], we have

$$\|f\|_{\mathrm{H}_1^c(\mathbb{R}^n;\mathcal{M})} \sim \inf\Big\{ \sum\nolimits_k |\lambda_k| \,\Big|\, f = \sum\nolimits_k \lambda_k a_k \text{ with } a_k \text{ column atoms}\Big\}.$$

On the other hand, we have already settled a dyadic filtration $(\mathcal{A}_k)_{k\in\mathbb{Z}}$ for our algebra of operator-valued functions $\mathcal{A}$. Then, we may follow [48] to define the corresponding noncommutative Hardy space $\mathrm{H}_1(\mathcal{A})$ as the completion of the space of finite martingales in $L_1(\mathcal{A})$ with respect to the norm

$$\|f\|_{\mathrm{H}_1(\mathcal{A})} = \inf_{\substack{f=g+h \\ g,h \text{ martingales}}} \Big\|\Big(\sum_{k\in\mathbb{Z}} dg_k dg_k^*\Big)^{\frac{1}{2}}\Big\|_1 + \Big\|\Big(\sum_{k\in\mathbb{Z}} dh_k^* dh_k\Big)^{\frac{1}{2}}\Big\|_1.$$

In other words, $\mathrm{H}_1(\mathcal{A}) = \mathrm{H}_1^r(\mathcal{A}) + \mathrm{H}_1^c(\mathcal{A})$ where the spaces on the right are the completions of the spaces of finite L_1-martingales with respect to the norms in L_1 of the corresponding row/column square functions given above. By the use of a dyadic covering [5, 34], it can be shown that there exists $n+1$ dyadic filtrations $\Sigma_{\mathcal{A}}^j$ $(0 \le j \le n)$ in $\mathbb{R}^n$ so that

$$\mathrm{H}_1(\mathbb{R}^n;\mathcal{M}) \simeq \sum_{j=0}^n \mathrm{H}_1(\mathcal{A}, \Sigma_{\mathcal{A}}^j),$$

where the latter spaces are defined as $\mathrm{H}_1(\mathcal{A})$ after replacing the standard filtration $\Sigma_{\mathcal{A}}^0$ by any other dyadic filtration in our family. Moreover, this isomorphism also holds independently for row/column Hardy spaces.

Proof of Theorem Aii). It suffices to show

$$T_r : \mathrm{H}_1^r(\mathcal{A}) \to L_1(\mathcal{A}) \quad \text{and} \quad T_c : \mathrm{H}_1^c(\mathcal{A}) \to L_1(\mathcal{A}),$$

for any generic noncommuting CZO (T_r, T_c). Indeed, in that case we decompose $f = f_r + f_c \in \mathrm{H}_1(\mathcal{A})$, so that $\|f\|_{\mathrm{H}_1(\mathcal{A})} \sim \|f_r\|_{\mathrm{H}_1^r(\mathcal{A})} + \|f_c\|_{\mathrm{H}_1^c(\mathcal{A})}$ and we deduce that

$$\|T_r f_r\|_1 + \|T_c f_c\|_1 \lesssim \|f_r\|_{\mathrm{H}_1^r(\mathcal{A})} + \|f_c\|_{\mathrm{H}_1^c(\mathcal{A})} \sim \|f\|_{\mathrm{H}_1(\mathcal{A})}.$$

According to our observation above, $\mathrm{H}_1(\mathcal{A})$ embeds isomorphically into $\mathrm{H}_1(\mathbb{R}^n;\mathcal{M})$ by means of a suitably choice of dyadic coverings of $\mathbb{R}^n$, and the same holds for row and column spaces isolatedly. Therefore, it also suffices to show that

$$T_r : \mathrm{H}_1^r(\mathbb{R}^n;\mathcal{M}) \to L_1(\mathcal{A}),$$

$$T_c : \mathrm{H}_1^c(\mathbb{R}^n;\mathcal{M}) \to L_1(\mathcal{A}).$$

Both estimates are identical, let us prove the column case. According to the atomic decomposition of $\mathrm{H}_1^c(\mathbb{R}^n;\mathcal{M})$ we just find a uniform upper estimate for the L_1 norm of $T_c(a)$ valid for an arbitrary column atom

$$\|T_c(a)\|_1 \le \big\|T_c(a)1_{2Q}\big\|_1 + \big\|T_c(a)1_{\mathbb{R}^n\setminus 2Q}\big\|_1.$$

The second term is dominated by

$$\begin{aligned}\big\|T_c(a)1_{\mathbb{R}^n\setminus 2Q}\big\|_1 &= \tau\int_{\mathbb{R}^n\setminus 2Q}\Big|\int_Q k(x,y)a(y)\,dy\Big|\,dx \\ &\le \int_Q\Big(\int_{\mathbb{R}^n\setminus 2Q}\big\|k(x,y)-k(x,c_Q)\big\|_{\mathcal{M}}\,dx\Big)\tau|a(y)|\,dy \\ &\lesssim \tau\Big(\int_Q|a(y)|\,dy\Big) \le \sqrt{|Q|}\tau\Big[\big(\int_Q|a(y)|^2\,dy\big)^{\frac12}\Big] \le 1,\end{aligned}$$

where the next to last estimate follows from Hansen's inequality or as a consequence of the operator-convexity of the function $a\mapsto|a|^2$. As for the first term, it suffices to show that $T_c: L_1(\mathcal{M};L_2^c(\mathbb{R}^n))\to L_1(\mathcal{M};L_2^c(\mathbb{R}^n))$, since then we find again

$$\begin{aligned}\big\|T_c(a)1_{2Q}\big\|_1 &= \tau\Big(\int_{2Q}|T_c(a)(x)|\,dx\Big) \\ &\le \sqrt{|2Q|}\,\tau\Big[\big(\int_{2Q}|T_c(a)(x)|^2\,dx\big)^{\frac12}\Big] \\ &\lesssim \sqrt{|2Q|}\,\tau\Big[\big(\int_Q|a(x)|^2\,dx\big)^{\frac12}\Big] \lesssim 1.\end{aligned}$$

The $L_1(\mathcal{M};L_2^c(\mathbb{R}^n))$-boundedness of T_c follows from anti-linear duality

$$\big\|T_c(f)\big\|_{L_1(\mathcal{M};L_2^c(\mathbb{R}^n))} \le \Big(\sup_{\|g\|_{L_\infty(L_2^c)}\le 1}\big\|T_c^*(g)\big\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}^n))}\Big)\|f\|_{L_1(\mathcal{M};L_2^c(\mathbb{R}^n))}.$$

It is easily checked that the adjoint $T_c^*(g)$ has the form $T_c^*g(x)\sim\int_{\mathbb{R}^n}k(y,x)^*g(y)\,dy$ when we construct it with respect to the anti-linear bracket $\langle f,g\rangle=\varphi(f*g)$. This means in particular that T_c^* is still an L_2-bounded column CZO associated to a kernel satisfying Hörmander smoothness. This gives rise to

$$\begin{aligned}\big\|T_c^*(g)\big\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}^n))} &= \Big\|\Big(\int_{\mathbb{R}^n}|T_c^*(g)(x)|^2\,dx\Big)^{\frac12}\Big\|_{\mathcal{M}} \\ &= \sup_{\|u\|_{L_2(\mathcal{M})}\le 1}\Big(\int_{\mathbb{R}^n}\big\langle|T_c^*(g)(x)|^2u,u\big\rangle_{L_2(\mathcal{M})}\,dx\Big)^{\frac12} \\ &= \sup_{\|u\|_{L_2(\mathcal{M})}\le 1}\Big(\int_{\mathbb{R}^n}\big\|T_c^*(gu)(x)\big\|_{L_2(\mathcal{M})}^2\,dx\Big)^{\frac12} \\ &\lesssim \sup_{\|u\|_{L_2(\mathcal{M})}\le 1}\Big(\int_{\mathbb{R}^n}\big\|g(x)u\big\|_{L_2(\mathcal{M})}^2\,dx\Big)^{\frac12} \\ &= \Big\|\Big(\int_{\mathbb{R}^n}|g(x)|^2\,dx\Big)^{\frac12}\Big\|_{\mathcal{M}}.\end{aligned}$$

The third identity above uses the right $\mathcal{M}$-module nature of column CZO's. $\square$

Remark 2.4. Theorem Aii) could have also been derived from the $L_\infty\to\mathrm{BMO}$ type estimates in [20]. We have preferred to include this alternative argument using atomic decompositions. Still a third approach is possible using more recent atomic decompositions from [2, 12]. This will be needed below for martingale transforms and paraproducts. The proof goes in fact a little further than the statement, since it emphasizes row/column $\mathrm{H}_1\to L_1$ type estimates for T_r/T_c respectively. This also works for arbitrary semicommutative CZO's under suitable assumptions, see [20] for details.

Remark 2.5. The proof above also shows that $L_1(L_2^\dagger)$ and $L_\infty(L_2^\dagger)$ boundedness of $T_\dagger$ for $\dagger \in \{r,c\}$ follow from the corresponding L_2 boundedness of the same operator. As noticed in [20], this is very specific of CZO's with noncommuting kernels since other semicommutative CZO's fail to satisfy this implication. The key property here is left/right $\mathcal{M}$-modularity, so that

$$uT_r(f) = T_r(uf) \quad \text{and} \quad T_c(f)u = T_c(fu).$$

This also explains our approach through weak type estimates, see the Appendix.

3.2.4 Row/column L_p estimates

Theorem B follows as an easy consequence of Theorem A after applying suitable interpolation/duality results. Thus, we will only outline the definition of the involved spaces and the necessary results to deduce Theorem B from Theorem A. Given $1 < p < \infty$, the noncommutative Hardy space $\mathrm{H}_p(\mathcal{A})$ is defined as

$$\mathrm{H}_p(\mathcal{A}) \ = \begin{cases} \mathrm{H}_p^r(\mathcal{A}) + \mathrm{H}_p^c(\mathcal{A}) & \text{if } 1 < p \leq 2, \\ \mathrm{H}_p^r(\mathcal{A}) \cap \mathrm{H}_p^c(\mathcal{A}) & \text{if } 2 \leq p < \infty, \end{cases}$$

where the corresponding row/column Hardy spaces arise as the completion of the subspace of finite martingales in $L_p(\mathcal{A})$ with respect to the norms given by the row and column square functions

$$\|f\|_{\mathrm{H}_p^r(\mathcal{A})} \ = \ \Big\| \Big(\sum_{k \in \mathbb{Z}} df_k df_k^* \Big)^{\frac12} \Big\|_p,$$
$$\|f\|_{\mathrm{H}_p^c(\mathcal{A})} \ = \ \Big\| \Big(\sum_{k \in \mathbb{Z}} df_k^* df_k \Big)^{\frac12} \Big\|_p.$$

Pisier/Xu obtained in [48] the noncommutative Burkholder-Gundy inequalities which can be formulated as $L_p(\mathcal{A}) \simeq \mathrm{H}_p(\mathcal{A})$ for $1 < p < \infty$. On the other hand, we know from [16, 25] that $\mathrm{H}_p^\dagger(\mathcal{A})^* \simeq \mathrm{H}_{p'}^\dagger(\mathcal{A})$ for $\dagger \in \{r,c\}$ and $1 < p < \infty$. Regarding interpolation, we know from Musat [**?**] that

$$\mathrm{H}_p^\dagger(\mathcal{A}) \ \simeq \ \big[\mathrm{H}_{p_0}^\dagger(\mathcal{A}), \mathrm{H}_{p_1}^\dagger(\mathcal{A}) \big]_\theta,$$

where $\dagger \in \{r,c\}$ and $\frac1p = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. The proof of Theorem B is now straightforward.

Proof of Theorem B. We know that

$$T_r : \mathrm{H}_1^r(\mathcal{A}) \to L_1(\mathcal{A}) \quad \text{and} \quad T_c : \mathrm{H}_1^c(\mathcal{A}) \to L_1(\mathcal{A}).$$

If $1 < p < 2$, we find $T_r : \mathrm{H}_p^r(\mathcal{A}) \to L_p(\mathcal{A})$ and $T_c : \mathrm{H}_p^c(\mathcal{A}) \to L_p(\mathcal{A})$ by interpolation with $L_2(\mathcal{A}) = \mathrm{H}_2^r(\mathcal{A}) = \mathrm{H}_2^c(\mathcal{A})$. Hence, taking a decomposition $f = f_r + f_c$ satisfying $\|f\|_p \sim \|f\|_{\mathrm{H}_p(\mathcal{A})} \sim \|f_r\|_{\mathrm{H}_p^r(\mathcal{A})} + \|f_c\|_{\mathrm{H}_p^c(\mathcal{A})}$ we get $\|T_r f_r\|_p + \|T_c f_c\|_p \lesssim \|f\|_p$. Now if $2 < p < \infty$, recalling that T_r^*, T_c^* are again row/column CZO's with the same properties, duality gives $T_r : L_p(\mathcal{A}) \to \mathrm{H}_p^r(\mathcal{A})$ and $T_c : L_p(\mathcal{A}) \to \mathrm{H}_p^c(\mathcal{A})$. This immediately yields the inequality in Theorem Bii). The $L_\infty \to \mathrm{BMO}$ type estimates were originally proved in [20], these also follows by duality from Theorem A. □

Remark 2.6. Alternatively, it can be proved that the row/column L_p estimates in Theorem Bi) for $1 < p < 2$ also follow by real interpolation from the weak type estimates in Theorem Ai). Moreover, since Mei's spaces $\mathrm{H}_p(\mathbb{R}^n; \mathcal{M})$ also behave well for interpolation and duality, the statement of Theorem B could have been done in terms of these other Hardy spaces.

3.3 Proof of Theorem C

In this section we turn our attention to noncommutative martingale transforms and paraproducts. In particular, the former pair $(\mathcal{A}, \varphi)$ will refer in what follows to an arbitrary semifinite von Neumann algebra equipped with a normal faithful semifinite trace. Our filtration $\Sigma_{\mathcal{A}} = (\mathcal{A}_k)_{k \geq 1}$ will be any increasing family of von Neumann subalgebras, whose union is weak-$*$ dense in $\mathcal{A}$. The operators E_k and Δ_k still denote the corresponding conditional expectations and martingale difference operators. As mentioned in the Introduction, we will deal with

a) Noncommuting martingale transforms

$$M_\xi^r f = \sum_{k \geq 1} \Delta_k(f) \xi_{k-1} \quad \text{and} \quad M_\xi^c f = \sum_{k \geq 1} \xi_{k-1} \Delta_k(f).$$

b) Paraproducts with noncommuting symbol

$$\Pi_\rho^r(f) = \sum_{k \geq 1} \mathsf{E}_{k-1}(f) \Delta_k(\rho) \quad \text{and} \quad \Pi_\rho^c(f) = \sum_{k \geq 1} \Delta_k(\rho) \mathsf{E}_{k-1}(f).$$

The martingale coefficients $\xi_k \in \mathcal{A}_k$ form an adapted sequence and it is easy to show that L_2-boundedness of M_ξ^r and M_ξ^c holds iff the ξ_k's are uniformly bounded in the norm of $\mathcal{A}$. On the other hand, the classical characterization $\Pi_\rho : L_2 \to L_2$ iff $\rho \in \mathrm{BMO}$ was disproved by Nazarov, Pisier, Treil and Volberg [41], see also Mei's paper [33]. Hence, the L_2-boundedness of Π_ρ^r and Π_ρ^c will be simply assumed in what follows. Regarding Cuculescu's construction and CZ decomposition, no essential changes are needed. Namely, given $f \in L_1^+(\mathcal{A})$ (the former space $\mathcal{A}_{c,+}$ is unnecessary since our filtration starts now at $k = 1$) and $\lambda \in \mathbb{R}_+$, Cuculescu's construction is verbatim the same. The only difference is on the diagonal estimate

$$\Big\| qfq + \sum_{k=1}^{\infty} p_k f_k p_k \Big\|_2^2 \lesssim \lambda \|f\|_1.$$

This inequality requires to work with regular filtrations, which are defined through the additional condition $\mathsf{E}_k(f) \leq c\,\mathsf{E}_{k-1}(f)$ for some absolute constant $c > 0$ and every pair $(f, k) \in \mathcal{A}_+ \times \mathbb{Z}_+$. Of course, the reader might think that it is more appropriate to use in this case the noncommutative form of Gundy's decomposition [44], which does not require any regularity assumption on the martingale. This leads unfortunately to some problems related to our triangular truncations which will be explained in the Appendix below.

Proof of Theorem C — Weak type inequalities. The argument is essentially the same as in the perfect dyadic case. Given $f \in L_1^+(\mathcal{A})$, we construct the same decomposition $f = f_r + f_c$ via the projections $\pi_{j,k}$ and fix $\lambda = 2^\ell$ for some $\ell \in \mathbb{Z}$. A further CZ decomposition gives $f_c = g_d^c + g_{\mathit{off}}^c + b_d^c + b_{\mathit{off}}^c$ as usual. According to our regularity assumption, we still have

$$\max \Big\{ \|g_d^r\|_2^2, \|g_d^c\|_2^2 \Big\} \leq \|g_d\|_2^2 = \Big\| qfq + \sum_{k \geq 1} p_k f_k p_k \Big\|_2^2 \lesssim \lambda \|f\|_1.$$

Thus, arguing as in the proof of Theorem A it suffices to show that

$$\widehat{q} M_\xi^r(\gamma^r) = M_\xi^c(\gamma^c) \widehat{q} = \widehat{q} \Pi_\rho^r(\gamma^r) = \Pi_\rho^c(\gamma^c) \widehat{q} = 0$$

for any $\gamma \in \{g_{off}, b_d, b_{off}\}$. As usual, we just consider the column case by symmetry. Let us begin with martingale transforms. Since $\gamma^c = \sum_j \mathsf{UT}_{j-1}(\Delta_j(\gamma))$ and the triangular truncation UT_{j-1} is built with j-predictable projections, we see that $\mathsf{UT}_{j-1}(\Delta_j(\gamma))$ is a j-th martingale difference, so that

$$\Delta_k(\gamma^c) \, = \, \mathsf{UT}_{k-1}(\Delta_k(\gamma)).$$

By the proof of Theorem A, we know $\mathsf{UT}_{k-1}(\Delta_k(\gamma))\widehat{q}_{k-1} = 0$ and

$$M_\xi^c(\gamma^c)\widehat{q} \, = \, \sum_{k=1}^\infty \xi_{k-1}\Delta_k(\gamma^c)\widehat{q} \, = \, \sum_{k=1}^\infty \xi_{k-1}\mathsf{UT}_{k-1}(\Delta_k(\gamma))\widehat{q}_{k-1}\widehat{q} \, = \, 0.$$

For martingale paraproducts, we observe that $\mathsf{E}_{k-1}(\gamma^c) = \sum_{j<k} \mathsf{UT}_{j-1}(\Delta_j(\gamma))$ and

$$\Pi_\rho^c(\gamma^c)\widehat{q} \, = \, \sum_{k=1}^\infty \Delta_k(\rho) \sum_{j<k} \mathsf{UT}_{j-1}(\Delta_j(\gamma))\widehat{q}_{j-1}\widehat{q} \, = \, 0. \qquad \square$$

Remark 3.1. Adjoints of martingale paraproducts have the form

$$\big[\Pi_\rho^c\big]^* f \, = \, \sum_{k\geq 1} \mathsf{E}_{k-1}\big(\Delta_k(\rho^*)\Delta_k(f)\big) \quad \text{and} \quad \big[\Pi_\rho^r\big]^* f \, = \, \sum_{k\geq 1} \mathsf{E}_{k-1}\big(\Delta_k(f)\Delta_k(\rho^*)\big)$$

when using the anti-linear duality bracket. It is easy to adapt the argument above for these maps, to obtain weak type inequalities for adjoints of noncommutative paraproducts associated to regular filtrations

$$\inf_{f=f_r+f_c} \big\| \big[\Pi_\rho^r\big]^* f_r \big\|_{1,\infty} + \big\| \big[\Pi_\rho^c\big]^* f_c \big\|_{1,\infty} \, \leq \, \|f\|_1.$$

We defined above the noncommutative Hardy spaces $\mathrm{H}_1(\mathcal{A})$. Alternatively, we may also consider the noncommutative form $\mathrm{h}_1(\mathcal{A}) \, = \, \mathrm{h}_1^r(\mathcal{A}) + \mathrm{h}_1^c(\mathcal{A}) + \mathrm{h}_1^d(\mathcal{A})$ of the conditional Hardy space h_1, where the norms are given by

$$\begin{aligned}
\|f\|_{\mathrm{h}_1^r(\mathcal{A})} &= \Big\| \Big(\sum_{k\geq 1} \mathsf{E}_{k-1}\big(df_k df_k^*\big) \Big)^{\frac{1}{2}} \Big\|_1, \\
\|f\|_{\mathrm{h}_1^c(\mathcal{A})} &= \Big\| \Big(\sum_{k\geq 1} \mathsf{E}_{k-1}\big(df_k^* df_k\big) \Big)^{\frac{1}{2}} \Big\|_1, \\
\|f\|_{\mathrm{h}_1^d(\mathcal{A})} &= \Big\| \sum_{k\geq 1} |df_k| \Big\|_1 \, = \, \sum_{k\geq 1} \|df_k\|_1.
\end{aligned}$$

The space $\mathrm{h}_1(\mathcal{A})$ was studied in [18, 45], it was independently proved that

$$\begin{aligned}
\mathrm{H}_1^r(\mathcal{A}) &\simeq \mathrm{h}_1^r(\mathcal{A}) + \mathrm{h}_1^d(\mathcal{A}), \\
\mathrm{H}_1^c(\mathcal{A}) &\simeq \mathrm{h}_1^c(\mathcal{A}) + \mathrm{h}_1^d(\mathcal{A}).
\end{aligned}$$

In conjunction, these isomorphisms could be regarded as a noncommutative form of Davis' decomposition for martingales. Shortly after, it was found in [2] an atomic decomposition for the spaces $\mathrm{h}_1^r(\mathcal{A})$ and $\mathrm{h}_1^c(\mathcal{A})$. More precisely, an element a in $L_1(\mathcal{A}) \cap L_2(\mathcal{A})$ is called a *column atom* with respect to the filtration $(\mathcal{A}_k)_{k\geq 1}$ if there exists $k_0 \in \mathbb{Z}_+$ and a finite projection $e \in \mathcal{A}_{k_0}$ such that

- $a = ae$,

- $\mathsf{E}_{k_0}(a) = 0$,
- $\|a\|_2 \leq \varphi(e)^{-\frac{1}{2}}$.

An element $a \in L_1(\mathcal{A})$ is called a $\mathrm{c}-\mathrm{atom}$ if it is a column atom or $a \in \mathcal{A}_1$ with $\|a\|_1 \leq 1$. Row atoms are defined to satisfy $a = ea$ instead and $\mathrm{r}-\mathrm{atoms}$ are defined similarly. We also refer to [12] for q-analogs of these notions. In the following result, we collect some norm equivalences coming from atomic decompositions and John-Nirenberg type inequalities. Recall that

$$\begin{aligned} \|f\|_{\mathrm{BMO}_c(\mathcal{A})} &= \sup_{k\geq 1} \Big\| \mathsf{E}_k \big[(f - f_{k-1})^*(f - f_{k-1}) \big] \Big\|_{\mathcal{A}}^{\frac{1}{2}}, \\ \|f\|_{\mathrm{bmo}_c(\mathcal{A})} &= \max \Big\{ \big\| \mathsf{E}_1(f) \big\|_1, \ \sup_{k\geq 1} \Big\| \mathsf{E}_k \big[(f - f_k)^*(f - f_k) \big] \Big\|_{\mathcal{A}}^{\frac{1}{2}} \Big\}. \end{aligned}$$

As usual, the corresponding row norms of f arise as the column norms of f^*. If we also define $\|f\|_{\mathrm{bmo}_d(\mathcal{A})} = \sup_k \|df_k\|_{\mathcal{A}}$, then we can define the spaces $\mathrm{BMO}(\mathcal{A})$ and $\mathrm{bmo}(\mathcal{A})$ as follows

$$\begin{aligned} \|f\|_{\mathrm{BMO}(\mathcal{A})} &= \max \Big\{ \|f\|_{\mathrm{BMO}_r(\mathcal{A})}, \|f\|_{\mathrm{BMO}_c(\mathcal{A})} \Big\}, \\ \|f\|_{\mathrm{bmo}(\mathcal{A})} &= \max \Big\{ \|f\|_{\mathrm{bmo}_r(\mathcal{A})}, \|f\|_{\mathrm{bmo}_c(\mathcal{A})}, \|f\|_{\mathrm{bmo}_d(\mathcal{A})} \Big\}. \end{aligned}$$

The isomorphism $\mathrm{BMO}(\mathcal{A}) \simeq \mathrm{bmo}(\mathcal{A})$ was independently proved in [18, 45].

Atoms and John-Nirenberg inequality [2, 12]. We have

$$\begin{aligned} \|f\|_{\mathrm{h}_1^r} &\sim \inf \Big\{ \sum\nolimits_k |\lambda_k| \ \big| \ f = \sum\nolimits_k \lambda_k a_k \text{ and } a_k \ \mathrm{r}-\mathrm{atom} \Big\}, \\ \|f\|_{\mathrm{h}_1^c} &\sim \inf \Big\{ \sum\nolimits_k |\lambda_k| \ \big| \ f = \sum\nolimits_k \lambda_k a_k \text{ and } a_k \ \mathrm{c}-\mathrm{atom} \Big\}, \\ \|f\|_{\mathrm{bmo}(\mathcal{A})} &\sim \sup_{k\geq 1} \Big[\|df_k\|_\infty \vee \sup_{\substack{\beta\in\mathcal{A}_k \\ \|\beta\|_1\leq 1}} \big\| \beta(f - f_k) \big\|_1 \vee \sup_{\substack{\beta\in\mathcal{A}_k \\ \|\beta\|_1\leq 1}} \big\| (f - f_k)\beta \big\|_1 \Big]. \end{aligned}$$

The last equivalence is a John-Nirenberg type inequality, which differs from [23].

Proof of Theorem C — H_p/L_p type inequalities. Let us begin with $\mathrm{H}_1 \to L_1$ type inequalities. As pointed out in the proof of Theorem Aii), it suffices to show that $T_\dagger : \mathrm{H}_1^\dagger(\mathcal{A}) \to L_1(\mathcal{A})$ with $\dagger \in \{r, c\}$ and for both martingale transforms and paraproducts. Since we have

$$\mathrm{H}_1^\dagger(\mathcal{A}) \simeq \mathrm{h}_1^\dagger(\mathcal{A}) + \mathrm{h}_1^d(\mathcal{A}),$$

it suffices to show that $T_\dagger : \mathrm{X} \to L_1(\mathcal{A})$ with X any of the two spaces appearing on the right. Once more, the argument is row/column symmetric and we just consider columns. To see that $T_c : \mathrm{h}_1^c(\mathcal{A}) \to L_1(\mathcal{A})$ we may use the atomic decomposition above, so that it suffices to find a uniform upper bound for $\|T_c(a)\|_1$ with a being a $\mathrm{c}-\mathrm{atom}$. If $a \in \mathcal{A}_1$ with $\|a\|_1 \leq 1$, then we see that

$$M_\xi^c(a) = \xi_0 a_1 \quad \text{and} \quad \Pi_\rho^c(a) = ba = \Pi_\rho^c(u|a|^{\frac{1}{2}})|a|^{\frac{1}{2}} \quad \text{for} \quad a = u|a|.$$

In particular, $\|M_\xi^c(a)\|_1 + \|\Pi_\rho^c(a)\|_1 \lesssim \|a\|_1 \leq 1$. If a is a column atom, we find

$$M_\xi^c(a) \ = \ \sum_{k>k_0} \xi_{k-1}\Delta_k(a) \ = \ \sum_{k>k_0} \xi_{k-1}\Delta_k(a)e \ = \ M_\xi^c(a)e,$$

$$\Pi_\rho^c(a) \ = \ \sum_{k>k_0+1} \Delta_k(\rho)\mathsf{E}_{k-1}(a) \ = \ \sum_{k>k_0+1} \Delta_k(\rho)\mathsf{E}_{k-1}(a)e \ = \ \Pi_\rho^c(a)e.$$

This gives rise to $\|T_c(a)\|_1 = \|T_c(a)e\|_1 \le \|T_c(a)\|_2\|e\|_2 \lesssim \|a\|_2\|e\|_2 \le 1$ for both martingale transforms and paraproducts. We have already justified the $\mathrm{h}_1^c \to L_1$ boundedness. Let us now look at h_1^d

$$\|M_\xi^c(f)\|_1 \ \le \ \sum_{k\ge1} \|\xi_k\|_\infty \|\Delta_k(f)\|_1 \ \le \ \Big(\sup_{k\ge1}\|\xi_k\|_\infty\Big)\|f\|_{\mathrm{h}_1^d(\mathcal{A})}$$

As for the paraproduct, we use the John-Nirenberg inequality above

$$\begin{aligned}\|\Pi_\rho^c(f)\|_1 \ &= \ \Big\|\sum_{k\ge1}\Delta_k(\rho)\sum_{j<k}\Delta_j(f)\Big\|_1 \\ &= \ \Big\|\sum_{k\ge1}\big(\rho-\rho_k\big)\Delta_k(f)\Big\|_1 \ \lesssim \ \|\rho\|_{\mathrm{bmo}(\mathcal{A})}\|f\|_{\mathrm{h}_1^d(\mathcal{A})}.\end{aligned}$$

According to [18, 45] and [33, 41], we have

$$\|\rho\|_{\mathrm{bmo}(\mathcal{A})} \ \sim \ \|\rho\|_{\mathrm{BMO}(\mathcal{A})} \ \lesssim \ \max\Big\{\big\|\Pi_\rho^r : L_2 \to L_2\big\|, \big\|\Pi_\rho^c : L_2 \to L_2\big\|\Big\}.$$

All together gives that M_ξ^c and Π_ρ^c take $\mathrm{H}_1^c(\mathcal{A})$ into $L_1(\mathcal{A})$ as we claimed. In fact slight modifications of the given argument yield the same result for $[\Pi_\rho^c]^*$, details are left to he reader. This is all what is needed to produce analog inequalities in this setting to those in Theorems A and B, we just need to follow the arguments verbatim. It remains to show that $\Pi_\rho^c : L_p(\mathcal{A}) \to L_p(\mathcal{A})$ for $p > 2$, for which it will be enough to prove $L_\infty \to \mathrm{BMO}$ boundedness and use interpolation. The $L_\infty \to \mathrm{BMO}_c$ boundedness follows by duality from the $\mathrm{H}_1^c \to L_1$ boundedness of $[\Pi_\rho^c]^*$. On the other hand, the $L_\infty \to \mathrm{BMO}_r$ boundedness is very simple

$$\begin{aligned}\|\Pi_\rho^c f\|_{\mathrm{BMO}_r(\mathcal{A})} \ &= \ \sup_{k\ge1}\Big\|\mathsf{E}_k\Big(\sum_{j\ge k}\Delta_j(\Pi_\rho^c(f))\Delta_j(\Pi_\rho^c(f))^*\Big)\Big\|_{\mathcal{A}}^{\frac12} \\ &= \ \sup_{k\ge1}\Big\|\mathsf{E}_k\Big(\sum_{j\ge k}\Delta_j(\rho)\mathsf{E}_{j-1}(f)\mathsf{E}_{j-1}(f)^*\Delta_j(\rho)^*\Big)\Big\|_{\mathcal{A}}^{\frac12} \\ &\le \ \sup_{k\ge1}\Big\|\mathsf{E}_k\Big(\sum_{j\ge k}\Delta_j(\rho)\Delta_j(\rho)^*\Big)\Big\|_{\mathcal{A}}^{\frac12}\|f\|_\infty \ \le \ \|\rho\|_{\mathrm{BMO}_r(\mathcal{A})}\|f\|_\infty.\end{aligned}$$

Now we majorize $\|\rho\|_{\mathrm{BMO}_r(\mathcal{A})}$ by the $L_2 \to L_2$ norm of Π_ρ as we did above. □

Observe that we have not needed to assume regularity of our martingale filtration and we find that $[\Pi_\rho^r]^*, [\Pi_\rho^c]^*$ take $\mathrm{H}_1 \to L_1$ and $L_p \to L_p$ for $1 < p < 2$ by duality. In some sense, row/column noncommutative paraproducts present a similar behavior as row/column square functions in the noncommutative Burkholder-Gundy and Khintchine inequalities [31, 32, 48]. On the other hand, [52, Theorem 5.7] yields $L\log L \to L_1$ type estimates for a finite von Neumann algebra $\mathcal{A}$ with (T_r, T_c) a martingale transform/paraproduct with noncommuting coefficients/symbol

$$\inf_{f=f_r+f_c}\big\|T_rf_r\big\|_1 + \big\|T_cf_c\big\|_1 \ \lesssim \ \|f\|_{L\log L(\mathcal{A})}.$$

3.4 Appendix. Open problems

A.1. CZO's with noncommuting kernels

Our proof of Theorem Ai) is not entirely satisfactory, since it does not include arbitrary CZO's with noncommuting kernels. In the general case, we can not expect to annihilate the terms associated to $g_{\mathit{off}}, b_d, b_{\mathit{off}}$. If the reader considers the simplest term b_d, a difficulty with triangular truncations in L_1 will be immediately recognized. In fact, our proof for Haar shifts operators does not provide sharp constants for the same reason.

Problem 1. Extend Theorem Ai) to arbitrary CZO's with noncommuting kernels.

Here is a possible alternative argument. Once we have $f = f_r + f_c$, the same decomposition constructed in the proof of the perfect dyadic case, we could consider a *left* CZ *decomposition* for f_r and a *right* CZ *decomposition* for f_c as follows. Given $\lambda \in \mathbb{R}_+$ we let $f_r = g_r + b_r$ and $f_c = g_c + b_c$ with

$$g_r \,=\, \widehat{q} f_r + \sum_{k \in \mathbb{Z}} \widehat{p}_k \mathsf{E}_k(f_r) \quad \text{and} \quad b_r \,=\, \sum_{k \in \mathbb{Z}} \widehat{p}_k \big(f_r - \mathsf{E}_k(f_r)\big),$$

where $\widehat{p}_k = \widehat{q}_{k-1} - \widehat{q}_k$. The column decomposition just requires to put $\widehat{p}_k$ and $\widehat{q}$ on the right. The advantage of this approach is that we do not find off-diagonal terms which were much harder to deal in [43]. Moreover, it is not very difficult to show that

$$\max \Big\{ \|g_r\|_2^2, \|g_c\|_2^2 \Big\} \,\lesssim\, \lambda \|f\|_1$$

as expected. Problem 1 would be solved if we knew that

$$\sum_{k \in \mathbb{Z}} \big\|\widehat{p}_k \big(f_r - \mathsf{E}_k(f_r)\big)\big\|_1 + \big\|\big(f_c - \mathsf{E}_k(f_c)\big)\widehat{p}_k\big\|_1 \,\lesssim\, \|f\|_1.$$

It is perhaps too optimistic to expect that the inequality above holds, since the triangular truncations LT_k and UT_k appear to be incomparable for different values of k. We wonder whether some noncommutative form of Davis' decomposition in the sense of [53] could be useful to modify our row/column decomposition $f = f_r + f_c$ before performing the CZ decomposition, see also [42] for related ideas. Note that such a row/column CZ decomposition would provide in particular a much simpler proof of the main result in [43], since off diagonal terms would disappear.

Problem 2. Find a row/column CZ decomposition of f in the line explained above.

A.2. CZO's on general von Neumann algebras

As explained in [43], a key ingredient for a successful application of the noncommutative CZ decomposition is to use it on $\mathcal{M}$-bimoludar maps. In this paper, our decomposition $f = f_r + f_c$ has allowed us to make it work for either left or right $\mathcal{M}$-module maps. There are however many other semicommutative CZO's, some of which were mentioned in the Introduction. We know from [20] that a semicommutative CZO satisfying $L_\infty(L_2^r)$ and $L_\infty(L_2^c)$ boundedness also satisfies $T : L_\infty(\mathcal{A}) \to \mathrm{BMO}(\mathcal{A})$.

Problem 3. Do we have $T : L_1(\mathcal{A}) \to L_{1,\infty}(\mathcal{A})$ under the same assumptions?

According to [20], solving Problem 3 for CZO's associated to a kernel acting by Schur multiplication would provide weak type $(1,1)$ inequalities for crossed product extensions of classical CZO's

$$Tf(x) \sim \sum_{g \in \mathrm{G}} \int_{\mathbb{R}^n} k(x,y) f_g(y) \rtimes_\gamma \lambda(g)\, dy$$

on $\mathcal{A} = L_\infty(\mathbb{R}^n) \rtimes_\gamma \mathrm{G}$. This in turn is closely related to weak type estimates for Fourier multipliers on group von Neumann algebras, see [20] for further details. On the other hand, consider CZO's of the form

$$Tf(x) \sim \int_{\mathbb{R}^n} (id \otimes \mathrm{tr}) \Big[k(x,y) \big(\mathbf{1} \otimes f(y) \big) \Big] \, dy.$$

As we have seen along this paper and in [43], weak type inequalities require to find vanishing products $q_1(y) q_2(x)$ with q_1, q_2 certain projections in $\mathcal{A}$, see e.g. Lemma 2.2. However, we find $T(fq_1)(x)q_2(x) \sim \int_{\mathbb{R}^n} (id \otimes \mathrm{tr})[k(x,y)(q_2(x) \otimes f q_1(y))] dy$ in the model above and no interaction between q_1 and q_2 takes place. This is due to the lack of right $\mathcal{M}$-modularity for T. In fact, solving Problem 3 for this kind of CZO's is very much related to the CZ theory for von Neumann algebras developed in [22]. Namely, the projection in Lemma 2.2 is a dyadic dilation on $\mathbb{R}^n$ of $\widehat{q}$ not affecting its $\mathcal{M}$ 'structure' because the CZO is given as a partial trace on $\mathbb{R}^n$, but not on $\mathcal{M}$. The idea in the model above is to dilate both in $\mathbb{R}^n$ and $\mathcal{M}$. Dilating in $\mathcal{M}$ has to do with finding a suitable 'metric' in $\mathcal{M}$ to work with. This is what is done in [22] in terms of diffusion semigroups on the given algebra. Under this point of view, we could relate CZO's on $(\mathcal{A}, \varphi)$ with those in [43] when φ is tracial and with the ones considered in this paper when φ is a nontracial weight.

Problem 4. Prove a CZ decomposition/weak type inequalities for CZO's in [22].

A.3. Gundy's decomposition vs triangular truncations

It is a little bit unsatisfactory to require regular filtrations to provide weak type inequalities for martingales transforms/paraproducts with noncommuting coefficients/symbols. It is well-known that these estimates hold in the classical setting for any filtration by means of Gundy's decomposition. The noncommutative extension of Gundy's decomposition was constructed in [44]. Given a positive martingale $f = (f_1, f_2, \ldots)$ in $L_1(\mathcal{A})$, we may decompose it as $f = \alpha + \beta + \gamma$ with

$$\begin{aligned} d\alpha_k &= q_k df_k q_k - \mathsf{E}_{k-1}\big(q_k df_k q_k\big), \\ d\beta_k &= q_{k-1} df_k q_{k-1} - q_k df_k q_k + \mathsf{E}_{k-1}\big(q_k df_k q_k\big), \\ d\gamma_k &= df_k - q_{k-1} df_k q_{k-1}. \end{aligned}$$

It was proved in [44] that

$$\max \Big\{ \frac{1}{\lambda} \|\alpha\|_2^2, \sum_{k \geq 1} \|d\beta_k\|_1, \lambda \varphi\Big(\bigvee_{k \geq 1} \mathrm{supp}^* d\gamma_k \Big) \Big\} \lesssim \|f\|_1,$$

where $\mathrm{supp}^* a = \mathbf{1}_{\mathcal{A}} - q$ with q the greatest projection satisfying $qaq = 0$. If we try to prove Theorem C using Gundy's decomposition instead of Calderón-Zygmund decomposition, we will not find any trouble controlling the terms associated to α and γ. The term β presents however a significant difficulty due to the presence of triangular truncations LT_k and UT_k in $L_1(\mathcal{A})$. This difficulty can be summarized as follows. Consider a family Tr_k of upper triangular truncations and assume that $(\alpha_k, \beta_k) \in L_\infty(\mathcal{A}) \times L_1(\mathcal{A})$, do we have

$$\Big\| \sum_{k \geq 1} \alpha_k \mathsf{Tr}_k(\beta_k) \Big\|_1 \lesssim \sum_{k \geq 1} \|\alpha_k \beta_k\|_1 ?$$

Or at least

$$\Big\| \sum_{k \geq 1} \alpha_k \mathsf{Tr}_k(\beta_k) \Big\|_1 \lesssim \Big(\sup_{k \geq 1} \|\alpha_k\|_\infty \Big) \sum_{k \geq 1} \|\beta_k\|_1 ?$$

The first condition suffices to manage paraproducts with noncommuting symbols, the second one is weaker but sufficient to deal with martingale transforms having noncommuting coefficients. When dealing with lower triangular truncations, we should have $\mathsf{Tr}_k(\beta_k)\alpha_k$ on the left and $\beta_k\alpha_k$ on the right hand side.

Problem 5. Does any of these inequalities hold?

Problem 6. Can we eliminate the regularity assumption from Theorem C?

Bibliographie

[1] P. Auscher, S. Hoffman, C. Muscalu, T. Tao and C. Thiele, Carleson measures tress, extrapolation, and T(b) theorems, Pub. Mat., 46 :257-325, 2002.

[2] T. Bekjan, Z. Chen, M. Perrin, Z. Yin, Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales. J. Funct. Analysis., 258(7) :2483-2505, 2010.

[3] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction, Springer, New York, 1976.

[4] O. Blasco, S. Pott, Embeddings between operator-valued dyadic BMO spaces, Illinois J. of Math., 52(3) :799-814, 2008.

[5] J. Conde, J. García-Cuerva and J. Parcet, On dyadic coverings and nondoubling Calderón-Zygmund theory, preprint 2012.

[6] I. Cuculescu, Martingales on von Neumann algebras. J. Multiv. Anal., 1 :17-27, 1971.

[7] A. Defant, M. Junge, Maximal theorems of Menchoff-Rademacher type in non-commutative L_q-spaces. J. Funct. Analysis., 206 :322-355, 2004.

[8] T. Fack and H. Kosaki, Generalized s-numbers of τ-measurable operators, Pacific J. Math., 123 :269-300, 1986.

[9] S.H. Ferguson and M.T. Lacey, A characterization of product BMO by commutators, Acta Math. 189(2) :143-160, 2002.

[10] C. Fefferman, E.M. Stein, Hp spaces of several variables, Acta Math., 129 :137-193, 1972.

[11] T. A. Gillespie, S. Pott, S. Treil, A. Volberg, Logarithmic growth for martingale transform, J. London Math. Soc, 64(3) :624-636, 2001.

[12] G. Hong and T. Mei, John-Nirenberg inequality and atomic decomposition for non-commutative martingales, preprint 2011, arXiv :1112.3187.

[13] T. Hytönen, Pseudo-localisation of singular integrals in L^p, Rev. Mat. Iberoam. 27(2) :557-584, 2011.

[14] T. Hytönen, The sharp weighted bound for general Calderón-Zygmund operators, to appear in Ann. Math.

[15] T. Hytönen, O. Salinas, and B. Viviani, Wavelet expansions for weighted vector-valued BMO functions, arXiv :0901.1577v1 [math.FA] 12 Jan 2009.

[16] M. Junge, Doob's Inequality for Non-commutative Martingales, J. Reine Angew. Math. 549 :149-190, 2002.

[17] M. Junge, C. Le Merdy and Q. Xu, H^∞ functional calculus and square functions on noncommutative Lp-spaces. Asterisque 305, 2006.

[18] M. Junge and T. Mei, Noncommutative Riesz transforms-A probabilistic approach, Amer. J. Math. 132(3) :611-680, 2010.

[19] M. Junge and T. Mei, BMO spaces assocated with semigroups of operators, Math. Ann., 2011.

[20] M. Junge, T. Mei and J. Parcet, Smooth Fourier multipliers on group von Neumann algebras, preprint 2011, arXiv Math : 1010.5320.

[21] M. Junge, T. Mei and J. Parcet, Transference in noncommutative harmonic analysis, in progress.

[22] M. Junge, T. Mei and J. Parcet, Algebraic Calderón-Zygmund theory, in progress.

[23] M. Junge, M. Musat, Non-commutative John-Nirenberg theorem, Trans. Amer. Math. Soc. 359(1) :115-142, 2007.

[24] M. Junge and D. Sherman, Noncommutative L_p modules, J. Operator Theory, 53 :3-34, 2005.

[25] M. Junge, Q. Xu, Non-commutative Burkholder/Rosenthal Inequalities, Ann. Prob. 31(2) :948-995, 2003.

[26] M. Junge and Q. Xu, On the best constants in some non-commutative martingale inequalities, Bull. London Math. Soc. 37 :243¨C253, 2005.

[27] M. Junge and Q. Xu, Noncommutative maximal ergodic theorems, J.Amer.Math.Soc. 20 :385-439, 2006.

[28] M. Junge and Q. Xu, Noncommutative Burkholder/Rosenthal inequalities II : applications, Israel J. Math., 167 :227-282, 2008.

[29] R.V. Kadison and J.R. Ringrose, Fundamentals of the Theory of Operator Algebras I and II, Grad. Stud. Math. 15 and 16, American Mathematical Society, 1997.

[30] N.H. Katz, Matrix valued paraproducts, J. Fourier Anal. Appl., 3 :913-921, 1997.

[31] F. Lust-Piquard, Inégalités de Khintchine dans C_p $(1 < p < \infty)$, C.R. Acad. Sci. Paris., 303 :289-292, 1986.

[32] F. Lust-Piquard and G. Pisier, Non-commutative Khintchine and Paley inequalities, Ark. Mat., 29 : 241-260, 1991.

[33] T. Mei, Notes on matrix valued paraproducts. Indiana Univ. Math. J. 55(2) :747–760, 2006.

[34] T. Mei, Operator valued Hardy spaces, Mem. Amer. Math. Soc. 188, 2007.

[35] T. Mei, Tent spaces associated with semigroups of operators, J. Funct. Anal. 255 :3356-3406, 2008.

[36] T. Mei, An extrapolation of operator-valued dyadic paraproducts, J. London Math. Soc., 81 :650-662, 2010.

[37] T. Mei and J. Parcet, Pseudo-localization of singular integrals and noncommutative Littlewood-Paley inequalities, Int. Math. Res. Not., 9 :1433-1487, 2009.

[38] Y. Meyer, Wavelets and Operators. Translated from the 1990 French original by D. H. Salinger. Cambridge Stud. Adv. Math., 37. Cambridge Univ. Press, Cambridge, 1992.

[39] C. Mucalu, J. Pipher, T. Tao, and C. Thiele, Bi-parameter paraproducts, Acta Math. 193 :269-296, 2004.

[40] M. Musat, Interpolation Between Non-commutative BMO and Non-commutative Lp-spaces, J. Funct. Analysis., 202(1) :195-225, 2003.

[41] F. Nazarov, G. Pisier, S. Treil, A. Volberg, Sharp Estimates in Vector Carleson Imbedding Theorem and for Vector Paraproducts, J. Reine Angew. Math., 542 :147-171, 2002.

[42] J. Parcet, Weak type estimates associated to Burkholder's martingale inequality, Rev. Mat. Iberoamericana., 23 :1-27, 2007.

[43] J. Parcet, Pseudo-localization of singular integrals and noncommutative Calderón-Zygmund theory, J. Funct. Anal., 256 :5509-593, 2009.

[44] J. Parcet, N. Randrianantoanina, Gundy's Decomposition for Non-Commutative Martingales and Applications, Proc. London Math. Soc., 93(1) :227-252, 2006.

[45] M. Perrin, A noncommutative Davis' decomposition for martingales, J. London Math. Soc., 80(3) :627-648, 2009.

[46] S. Petermichl, Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol, C. R. Acad. Sci. Paris., 330 :455-460, 2000.

[47] G. Pisier, Non-commutative Vector Valued Lp-Spaces and Completely p-Summing Maps, Soc. Math. France. Astérisque 237, 1998.

[48] G. Pisier, Q. Xu, Non-commutative Martingale Inequalities, Comm. Math. Phys. 189 :667-698, 1997.

[49] G. Pisier, and Q. Xu, Non-commutative L_p-spaces, pp. 1459-1517 in "Handbook of the Geometry of Banach Spaces", Vol. II, edited by W.B. Johnson and J. Lindenstrauss, Elsevier, 2003.

[50] S. Pott and M.P. Smith, Paraproducts and Hankel operators of Schatten class via p-John-Nirenberg theorem, J. Funct. Anal., 217 :38-78, 2004.

[51] N. Randrianantoanina, Non-commutative martingale transform, J. Funct. Analysis., 194(1) :181-212, 2002.

[52] N. Randrianantoanina, A weak type inequality for non-commutative martingales and applications, Proc. London Math. Soc., 91 :509-544, 2005.

[53] N. Randrianantoanina. Conditioned square functions for noncommutative martingales. Ann. Probab., 35(3) :1039-1070, 2007.

[54] E. Ricard, L'espace H^1 n'a pas de base complètement inconditionnelle, C. R. Acad. Sci. Paris 331 (2000), 625-628.

[55] E. Ricard, Décomposition de H^1, Multiplicateurs de Schur et Espaces d'Operateurs, Thèse de Doctorat de l'Université Paris VI. 2001.

[56] M. Takesaki, Theory of operator algebras I, II and III, Encyclopedia of Mathematical Sciences 124, 125 and 127, Springer-Verlag, Berlin, 2003.

[57] F. Weisz, Atomic Hardy spaces, Anal. Math., 20 :65-80, 1994.

[58] T. Wolff, A note on Interpolation Spaces, Harmonic Analysis, Lec. Notes in Math., 1568, Spring-Verlag, Berlin-Heidelberg-New York, 1994.